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BOURDON'S

ARITHMETIC:

CONTAINING

A DISCUSSION OF THE THEORY OF NUMBERS.

TRANSLATED FROM THE FRENCH OF M. BOURDON, AND ADAPTED
TO THE USE OF THE COLLEGES AND ACADEMIES
OF THE UNITED STATES,

BY
CHARLES S. VENABLE,
LICENTIATE INSTRUCTOR IN THE UNIVERSITY OF VIRGINIA; FORMER PROFESSOR OF MATHEMATICS
IN HAMPDEN SIDNEY COLLEGE, VIRGINIA; FORMER PROFESSOR OF NATURAL
PHILOSOPHY AND CHEMISTRY IN THE UNIVERSITY OF GEORGIA.

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PREFACE.

I AM led to offer the present translation to the public, from the conviction that such a work is very much needed in our Academies and Colleges. In fact, a long experience in teaching has convinced me that, one great difficulty which the young student has to encounter in the study of Algebra and the higher branches of analysis, results from the want of sound philosophical ideas on the fundamental properties of numbers, and from the fact that the fundamental operations of Arithmetic are generally learned by rote, and not pursued as a system of close reasoning. Bourdon's treatise is the one adopted in the schedule of public instruction by the University of France. In preparing the translation, I have compared the seventh with the twenty-ninth Paris edition, and endeavoured to select the best methods of each. In this selection and arrangement, I have followed the outline of the lectures upon Arithmetic, delivered by the late Professors Bonnycastle and Courtenay, in the University of Virginia. The tables have been re-arranged, and a collection of examples annexed to the work.

The portions of Bourdon's very complete treatise on the Extraction of Roots, Progressions, Logarithms, and their applications, I have left out, because they are very thoroughly discussed in the best treatises on Algebra adopted by our Colleges and Universities. I have followed the author in introducing some few of the signs and preliminary definitions of Algebra. This usage the author well defends, as follows: — "To attempt to make known even some of the simple properties of numbers without employing the signs of algebra, is to present them in a manner very incomplete and little methodical. To use these signs to some extent, enables us to establish the connexion between these properties and their most important applications. Moreover, the discussion of these properties, a knowledge of which is essential to a thorough knowledge of arithmetic, cannot properly enter into the elements of algebra, without breaking the chain of theories which constitute this other branch of mathematics. In fine, the work is designed for those who wish to make the first steps in the career of a scientific or liberal education in a sure and profitable manner." The translator hopes the present treatise will be a useful addition to the means of thorough instruction in the United States.

C. S. V.

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Signs made use of in the Work.

1st. + *plus,* the sign of addition.
2d. — *minus,* the sign of subtraction.
3d. × *multiplied by,* the sign of multiplication.
4th. ÷ *divided by,* “ “ division.
5th. = *equal to,* “ “ equality.

ELEMENTS OF ARITHMETIC.

FIRST PART.

INTRODUCTION.

1. We call magnitude, or quantity, every thing which admits of increase or diminution. For example, lines, surfaces, solids, intervals of time, weights, are magnitudes. We can only form an exact idea of a magnitude by comparing it with another magnitude of the same species, and this second magnitude is called unity, in as much as it is to serve as a term of comparison for all magnitudes of the same species. Thus, when we say that a wall is twenty yards long, we are understood to have already acquired the idea of the unit of length called yard, and we suppose that, after having laid down the yard twenty times upon the length of the wall, we have arrived at the end.

Unity, in mathematics, *is then a magnitude of any species whatever, taken arbitrarily or in nature, which serves as a term of comparison for all magnitudes of the same species.* Whence it follows that there are as many species of units as of magnitudes.

The result of the comparison of any magnitude whatever with its unit, is called *number.* A number is called *entire* when it is the assemblage of several units of the same species or denomination. Thus, twenty dollars, thirty pounds, eight, twelve, fifteen units, of any species whatever, are *entire* numbers.

A *fraction* is a part of a unit.

A *fractional* or *mixed* number is an assemblage of several units of the same denomination, and of parts of this unit.

2. When, in enunciating a number, we add at the end of that number the species of magnitude taken for the unit, the number is called *concrete*. Thus, five feet, fifteen hours, six leagues, are *concrete* numbers. The first time we pronounce a number, the only sense we can attach to it, is the representing to ourselves a unit of a certain denomination, to which we compare another magnitude of the same denomination. But, by degrees, the mind, which accustoms itself to abstractions, represents to itself a collection of any like objects, of which each one is unity. In this case the collection is called an *abstract number*, because, in enunciating it, we make abstraction of the species of unit to which we refer it. It is in this last light that we are to consider numbers, in the discussion of the methods relating to the different operations which we have to perform upon them, if we wish to establish these methods so as to be able to apply them to all possible questions.

NUMERATION.

3. The first researches on numbers should have, necessarily, for object, the giving them names easy to retain; and, as there exists an infinity of numbers (since we can add to any number whatever, already formed, a new unit, which gives rise to a new number, also capable of being augmented by unity), it is necessary to find some means of expressing all numbers by a limited number of words, combined together in fit manner. Such is the object of *spoken numeration.*

Again, each one of the words which enter into the nomenclature of numbers being expressed by several letters, it was found necessary to invent an abridged mode of writing these words and their combinations, in order that the mind might be able to seize with more facility the reasonings which we are obliged to make upon the numbers. This is the object of *written numeration*, which consists in representing, by a limited number of characters or ciphers, the numbers enunciated in the ordinary language.

4. *Spoken Numeration.* — Though the nomenclature of entire numbers is known, for the most part, to the young men for whom these elements are written, we think it best to give a succinct, yet methodical analysis of it; for, the numeration which is adopted in nearly all countries, is founded upon this nomenclature.

The first numbers are, *one, two, three, four, five, six, seven, eight, nine.* These numbers are called simple units, or, *units* of the *first order.* Adding a new unit to the number *nine*, we form the number *ten*, which we regard as a new denomination, or, species of unit called *a ten*, or, *a unit of the second order.* We count by tens in the same manner as we have counted by simple units. Thus, we say, one *ten*, two *tens*, &c., &c.; *ten, twenty, thirty, forty, fifty, sixty*, &c. Between ten and twenty there are nine other numbers, which in English have the names, *eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen;* names established by usage, showing by their derivation, the addition of the preceding simple units successively to the unit of the second order.

Between twenty and thirty, there are also nine numbers, which are enunciated, *twenty-one, twenty-two*, &c. And thus we can enunciate all the numbers up to *ninety-nine.* This last number, augmented by *one*, gives *ten tens*, or the number *one hundred*, which we regard as a new unit, or unit of the *third order;* and we count by hundreds as we have counted by units and tens. Thus, *one hundred, two hundred*, &c. Placing successively between the words *hundred* and *two hundred, two hundred* and *three hundred, eight hundred* and *nine hundred*, and, after *nine hundred*, all the numbers comprised between one and ninety-nine, we form the names of all the numbers, from one hundred to nine hundred and ninety-nine. We can see that, in the enunciation of all these numbers, we have employed only the generic terms, *one, two, three, four, five, six, seven, eight, nine, ten, hundred,* and words easily derivable from these.

Adding *one* to the number *nine hundred and ninety-nine,* we obtain a collection of *ten hundreds,* or the number *thousand,* which forms the *unit of the fourth order.* Having reached this number it is agreed, in order not to multiply words too much, to regard *thousand* as a new principal unit, before the name of which we place the names of the nine hundred and ninety-nine first numbers. Thus, we say, *one thousand, two thousand, nine hundred and ninety-nine thousand.* A *ten thousand* forms the *unit of the fifth order;* a *hundred thousand* forms the *unit of the sixth order.*

Now, placing between two consecutive numbers of the denomination *thousand,* as *twenty* thousand and *twenty-one* thousand, the names of all the numbers of lower denomination than thousands, it is clear that we can thus enunciate all the numbers up to *nine hundred and ninety-nine thousand, nine hundred and ninety-nine.* This last number, augmented by *one,* gives *ten hundred thousand,* or, a *thousand thousand,* to which collection the name *million* has been given; in the same manner the collection of *thousand millions* is called *billions;* the collection of *thousand billions* is called *trillions,* and so on to infinity.

We count by *millions, billions,* and *trillions,* as we have counted by *thousands;* and it is easy to see that, by joining to the generic words indicated above, the words *million, billion, trillion, quatrillion, quintillion,* we will form the nomenclature of all imaginable entire numbers. We observe, in order to terminate this part of the subject, that the million is the *unit of the seventh order,* ten millions are *units of the eighth order,* hundred millions, *units of the ninth order.*

5. *Written Numeration.*—Though the above nomenclature is very simple, still we would find much trouble in combining together two or more large numbers, unless we had some abridged mode of writing them. This is easily arrived at by reflecting a little upon the nomenclature. We observe at once, that, among the words employed to express numbers, the one part, as *one, ten,*

hundred, thousand, ten thousand, &c., express the units of different orders, while the words, *one, two, three, nine*, express how many times each of these sorts of units enter into a number. This being established, if we agree to represent the first nine numbers by the characters or ciphers,

1	2	3	4	5	6	7	8	9
one,	*two,*	*three,*	*four,*	*five,*	*six,*	*seven,*	*eight,*	*nine,*

the whole difficulty consists in finding a means of making these ciphers express the different orders of unity which compose the proposed number. Then, establishing this principle (purely conventional), *that every figure placed to the left of another, expresses units of the order next higher to those of the other figure*, or, in other words, that *when several characters, signifying the first nine numbers, are written one after another*, then the first figure to the right expresses *simple units, the next on the left, tens, the third figure counting from right to left, hundreds, the fourth, thousands;* it is easy to see that, in general, we can represent all numbers by the aid of the preceding characters.

Character 0. — While this is true in general, nevertheless, there are numbers which the preceding convention fails to represent, unless we agree to use an additional character. If we undertake to write in figures the numbers, *ten*, *twenty*, *thirty*, &c., these numbers containing no simple units, we are compelled to adopt a *character which has no value by itself*, but which serves to hold the place of the units of the order which is wanting in the number enunciated. This cipher is 0, and is called *zero*. By the aid of this *cipher*, the numbers, ten, twenty, &c., are expressed by 10, 20, 30, 40, &c.

In the same manner, the numbers, *one hundred, two hundred*, &c., which contain neither simple units nor tens, are written thus:

100, 200, 300.

In general, the *zero* is a cipher which has no value by itself, but which we employ to hold the place of the different orders of unity which may be wanting in the number to be written. The other characters are called *significant figures,* and have two values; the one we call absolute, and is no other than that of the figure itself considered alone; the other, we call relative, which the figure acquires according to the place which it occupies to the left of other figures.

Now, if we reflect that every number is composed of simple units, of tens, of hundreds, &c.; that the collection of units of each order is equal to *nine;* that, in the case where a number is deprived of certain orders of units, we have a character to hold their places, we will see at once that there is no entire number which cannot be expressed by the aid of a certain combination of the ten characters: —

1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

Take the example, thirty-six billions, five hundred millions, twenty thousand, four hundred and seven.

This number contains seven simple units, no tens; four hundreds, no ones of thousands; two tens of thousands, no hundreds of thousands; no ones of millions, no tens of millions, five hundreds of millions; six ones of billions, and three tens of billions; then the number will be represented by 365000 20 407.

The system of numeration which we have just explained, has received the name of the decimal system, because we employ ten figures to express all numbers. *Ten,* or the number of characters employed is called *the base* of the system.

6. Let us make, now, an important observation: it results from the nomenclature, that every number can be divided into hundreds, tens, and simple units; into hundreds, tens, and ones of thousands; into hundreds, tens, and ones of millions, etc.; that is to say, into sets of simple units, thousands, millions, &c., each set expressed by three figures, except the last, which is that of the units of the highest order, and which cannot have more than two figures, and sometimes contains only one. When, then, we have become familiar with the manner of writing the numbers of three figures, it is sufficient to write, successively, the different

sets one to the left of the other; the set of units, the set of thousands, the set of millions, &c. We can even commence at the left; that is to say, write first the set of the units of the highest denomination, and, to the right of this, the other sets in the order of the magnitude of their units. It is thus that we ought to write a number *dictated* in ordinary language. But it is necessary to take care not to omit the zeros destined to replace the orders of units which are wanting; and there can never be any difficulty, since we know that each set, except the first to the left, must always contain three figures. Suppose, for example, that we have to write, by aid of our characters, *four hundred and six billions, twenty-eight millions, two hundred and fifty thousand, and forty-eight.*

Write in succession, each to the right of the other, the period of *billions;* the period of *millions;* the period of *thousands;* and, lastly, that of *simple units;* we will have 406, 028, 250, 048.

7. It is upon the preceding observation that the following means of translating into ordinary language, any number whatever written in figures, is founded.

After having separated the number into periods of three figures each, commencing at the right, *enunciate successively each period, setting out from the first period on the left,* and taking care to give to each period the name which belongs to it.

Example: 70345601. This number, being divided 70,345,601, is composed of seventy millions, three hundred and forty-five thousand, six hundred and one.

8. It remains for us still, in order to complete the theory of enumeration, to show the mode of writing fractions by means of figures. But we must first give a clear and precise idea of fractions, such as we consider them in arithmetic.

Let us suppose that we have to determine the length of a piece of cloth. Taking the unit of length called *yard,* and applying it as many times as possible to the length of the piece, two cases may occur, either, after the unit has been applied a certain number of times—15 times, for example, nothing will remain—or, we will obtain a remainder less than the yard. In the first case, the piece will contain *an entire number* of yards. In the second case, it will be necessary in order to have the whole length of the piece, to add to these 15 yards the fraction or part of the yard which remains. But how value this part? how compare it to the unit? We can first conceive this unit separated into two equal parts or *halves;* and, if the remainder is exactly equal to one of these halves, we say that the piece of cloth is 15 yards and *one half* long.

If the remainder is less or greater than a half yard, we conceive this *half* divided into *two* new equal parts, called quarters.

Instead of dividing the unit into *two* or *four* equal parts, we can conceive it to be divided into three equal parts called *thirds,* &c., &c.

Whence, we see that in order to form a clear idea of a fraction of a unit of any denomination whatever, it is necessary to conceive that this unit be divided into a certain entire number of equal parts, and that we take one, two, three, &c., of these parts; these parts thus taken, constitute what is called a *fraction.* Thus, the enunciation of a fraction involves necessarily two entire numbers, to wit:—*that which denotes into how many parts the unit has been divided,* called the *denominator; and that which denotes how many of these parts are necessary to form the fraction,* called the *numerator.* For example, five-eighths of a yard, thirteen-twentieths of a pound, are fractions. In the first, we conceive the yard divided into eight parts, and that we take five of these parts to form the fraction, *eight* is the denominator, and *five* the numerator. . . . (We see that in the spoken numeration of fractions the numerator remains unchanged in name, while the denominator is generally changed by the addition of *th.*)

It results, also, from the above, that a fraction is a magnitude referred to a part of the principal unit, which part we can consider itself as a particular species of unit. Thus, the fraction *thirteen-twentieths* of a yard, being composed of *thirteen* times the *twentieth* of a yard, this *twentieth* is a particular unit, which the proposed fraction contains *thirteen* times. This being established, two fractions are said to be of the same species when their denominator is the same, (the original or compound unit being likewise the same). For example, five-twelfths and seven-twelfths of a yard are fractions of the same species; but three-fourths and two-thirds of a pound are fractions of different species or denominations, because the denominators are different.

In order to express a fraction in figures, we place the numerator above the denominator, with a line between. Thus, the fraction three-fourths is denoted by $\frac{3}{4}$, seven-twelfths by $\frac{7}{12}$.

Reciprocally, $\frac{7}{8}$, $\frac{13}{15}$, represent the fractions, seven-eighths, thirteenth-fifteenths, that is to say, we enunciate the numerator, and then the denominator, and add the termination—*th*, to the latter.

CHAPTER I.

9. ARITHMETIC has, for it special object, to establish fixed and and certain rules for performing all possible operations upon numbers. It embraces, besides, the study of a great number of properties which have been discovered during the researches made in order to arrive at these methods, or to facilitate the use of them. We will now explain these operations in their order, recollecting that, in order to render the methods independent of every sort of question, it is best to consider the numbers as *abstract numbers.*

Nevertheless, in the applications designed to familiarize beginners with the methods, we can propose questions also relating to *concrete or denominate numbers.*

OPERATIONS ON ENTIRE NUMBERS.

ADDITION.

10. *To add or sum up several numbers, is to unite all these numbers into a single one; or, to form a number which contains in itself alone as many units as there are in the different numbers taken separately.*

The result of this operation is called the *sum*, or *total.* The addition of numbers of a single figure offers no difficulty. It is done unit by unit. Children learn thus to make these additions by means of their fingers, and fix the results in their memory. In this way, for example, they find thirty to be the sum of 5, 7, 8, 4, and 6; or, that 42 is the sum of the numbers 7, 9, 6, 5, 8, 7.

OPERATION.

$$\begin{array}{r} 7{,}453 \\ 1{,}534 \\ \hline 8{,}987 \end{array}$$

Let now the numbers to be added be 7,453 *and* 1,534.

After having written the numbers, one under another, with a line under them, we commence with the simple units, and say, 3 and 4 make 7, which we place under the units. Passing to the tens, 5 and 3 make 8, which we write under the tens. Then, 4 and 5 make 9, which we write under the hundreds. Lastly, 7 and 1 make 8, which we write in the column of thousands.

The number, 8,987, found by this operation, is the sum of the two given numbers, since it contains their units, their tens, their hundreds, and their thousands, which we have summed up successively.

OPERATION.

$$\begin{array}{r} 5{,}047 \\ 859 \\ 3{,}507 \\ 846 \\ \hline 10{,}259 \end{array}$$

Again, let it be proposed to add the four numbers, 5,047, 859, 3,507, 846. We write them one under the other, and, commencing with the units, 7 and 9 make 16, and 7 make 23, and 6 make 29. We place the nine simple units under the first column, and retain the two tens, in order to add them to the figures of the next column, which are also tens. Passing to the next column, we say that the two reserved, and 4 make 6, and 5 make 11, and 0 make 11, and 4 make 15. We write the 5 in the column of tens, and retain the 1 hundred which we carry to the column of hundreds. Operating upon this column, as upon the preceding, we find 22 hundreds, or 2 hundreds, which we write under the hundreds, and 2 thousands, which we retain in order to carry them to the column of thousands. Lastly, 2 reserved, and 5 make 7, and 3 make 10. We place the 0 under the thousands and advance the 1 to the left, which gives 10,259 for the required sum.

General Rule. — *In order to add several numbers together, commence by writing them one under another, so that the units of the same order may be in the same column. Add then successively the figures which compose each one of the vertical columns, commencing with the column of simple units, passing to the columns which are on the left: write below the line the sum of the figures of each column, provided the sum is expressed by a single figure. But if it exceeds 9, in which case it is expressed by several figures, of which the last to the right represents the units of this column, and the others to the left tens of the same order, write only the figure of units below the column, and reserve the tens in order to add them to the figures of the column immediately to the left. When you have operated in this manner upon all the columns, you will obtain the sum required, because it results from the union of the units, tens, hundreds, &c., which enter into the given numbers.*

11. *Remark.*—If the sum of the figures in each column does not exceed nine, we could commence the operation equally well by the addition of the units of the highest order as by the addition of the simple units. But as it happens oftenest that several of these sums exceed nine, if we commence on the left, we will often be obliged to return upon our steps, in order to correct a figure already written, and increase it by as many units as we shall have obtained from the tens of the following column in operating upon that column. For this reason it is best in all cases to commence on the right rather than on the left.

SUBTRACTION.

12. *To subtract one number from another is to seek the excess of the greater number over the less.* The result of this operation is called *remainder*, *excess*, or *difference.* So long as the numbers proposed consist only of a single figure, the subtraction is easy

Thus, the difference between 9 and 6 is 3. We can easily subtract a number of a single figure from a number which does not exceed twenty. Thus, take 7 from 13, there remains 6, since by what we have learned in addition, 7 and 6 make 13. In the same manner, 9 from 17 there remains 8, because 8 and 9 make 17. These operations, which suppose only the exercise of the memory upon the addition of numbers of a single figure, serve as a basis for the subtraction of numbers of several figures.

Let it be required to subtract 5467 *from* 8789.

OPERATION.

$$\begin{array}{r}8789\\5467\\\hline 3322\end{array}$$

After having placed the smaller number under the greater, and underlined the whole, we say, commencing with the simple units, 7 from 9 leave 2, which we place in the column of simple units; passing to the tens, 6 from 8 leave 2, which we write in the column of tens; the same operation finally upon the hundreds and thousands, 4 from 7 leave 3, and 5 from 8 leave 3, gives 3322 for the *required remainder.* For by the nature of the operations which have just been performed, we see that the greater number contains more than the second, 2 simple units, 2 tens, 3 hundreds, 3 thousands, and consequently exceeds the smaller by 3322.

Let us propose for a second example, *to find the difference which exists between the two numbers,* 83456 *and* 28784.

OPERATION.

$$\begin{array}{r}83456\\28784\\\hline 54672\end{array}$$

Having arranged the numbers as in the preceding example, we say, first, 4 from 6 leave 2, which we write under the units. But when we pass to the column of tens, we meet with a difficulty: the lower figure, 8, is greater than the upper one, 5, and consequently cannot be subtracted. In order to overcome this difficulty, we borrow mentally from the hundreds figure 1 hundred, which equals 10 tens, and add it to the 5 tens which we have already, giving us 15 tens; we then say, 8 from 15 leave 7, which we write in the column of tens. Passing to the column of hundreds, we observe that the upper figure, 4, ought to be diminished by 1, since we have borrowed this unit in the preceding subtraction; we say, then, 7 from 3, which is impossible; but we borrow, as before, 1 thousand, which equals ten hundreds, giving 13 hundreds, and take 7 from 13, which gives 6, to be written in the column of hundreds. Passing to the thousands, 8 cannot be taken from 2; but 8 from 12 leave 4, to be written in the column of thousands. Lastly, as the figure 8, of tens of thousands, on account of the 1 just borrowed, ought to be replaced by 7, we say, 2 from 7 leave 5. Thus, the *remainder*, or the excess of the greater number over the less, is 54672.

In order to understand how, by this means, we arrive at the end proposed, it is sufficient to remark that, according to the artifices employed to effect the partial subtractions, we can arrange the two numbers in the following manner:—

	Tens of thousands,	thousands,	hundreds,	tens,	units.
1st number,	7	12	13	15	6
2d number,	2	8	7	8	4
	5	4	6	7	2

From this we see that the upper number exceeds the lower one by two units, 7 tens, 6 hundreds, 4 thousands, and 5 tens of thousands — or exceeds it 54672 units.

Let it be proposed, for example, to subtract 158429 *from* 300405.

OPERATION.

$$\begin{array}{cccccc} & 9 & 9 & & 9 & \\ 3 & 0 & 0 & 4 & 0 & 5 \\ 1 & 5 & 8 & 4 & 2 & 9 \\ \hline 1 & 4 & 1 & 9 & 7 & 6 \end{array}$$

As 9, the units figure of the lower number, is larger than 5, the corresponding figure of the greater, we have to borrow 1 ten from the first figure to the left; but this figure being 0, it is necessary to have recourse to the figure 4, of hundreds, from which we borrow 1, which equals 10 tens; and since we have need of only a single ten, we leave 9 of them above the 0; we then add 1 ten to 5, which gives 15, and say, 9 from 15 leave 6, which we write under the units. Passing to the tens, we say, 2 from 9 leave 7.

For the hundreds, as the upper figure, 4, has been diminished by the 1 which we borrowed, and as we cannot take 4 from 3, we have recourse to the next figure to the left; but that and the figure which is to its left being zeros, we borrow 1 from the next significant figure, 3. This 1 equals 10 of the order following, and 100 units of the order thousands; and since we have need of only 1 unit of this order, we leave 99 of them, which we place above the two zeros; adding 1 thousand to the 3 hundreds, it becomes 13 hundreds, and we say, 4 from 13 leave 9, which we place under the column of hundreds.

In the two following subtractions, each one of the zeros being replaced by a 9, we say, 8 from 9 leave 1, and 5 from 9 leave 4. Passing to the first column to the left, we say, 1 from 2 (for the figure 3 is diminished by 1) leaves 1. Thus we have for the required remainder 141976.

If we reflect upon the manner in which the greater number has been decomposed, we can arrange the operation thus:—

	hundreds of thous.,	**tens of thous.,**	**thous.,**	**hundreds,**	**tens,**	**units.**
1st number,	2	9	9	13	9	15
2d number,	1	5	8	4	2	9
	1	4	1	9	7	6

Then the greater number exceeds the less by 6 units, 7 tens, 9 hundreds, 1 thousand, 4 tens of thousands, 1 hundred thousand, or by 141976.

General Rule.—*In order to perform the subtraction of two numbers, place the less number under the greater, so that the units of the same denomination fall in the same column; then underline the two numbers; subtract then successively, units from units, tens from tens, hundreds from hundreds, &c., and write the partial remainders one to the left of another; the number formed by these remainders is the total remainder, or the result required.*

When a figure of the lower line is greater than the figure above it, augment mentally this last figure by 10 *units, and diminish the figure to the left of it by one unit.*

If immediately to the left of an upper figure less than the one below, corresponding, there are one or more zeros, increase this figure above mentally always by 10 *units; but in the following subtractions replace the* 0s *by* 9s, *and diminish by a unit the upper significant figure which is immediately to the left of these zeros.*

13. *First Remark.* — If each one of the figures of the lower number is less than the corresponding figure of the greater, we could commence the operation indifferently at the right or left. But as it often happens that one of the figures of the less number exceeds the figure of the greater above it, the partial subtraction cannot be effected without borrowing from one of the figures to the left of that one with which we are operating; for this reason it is necessary to commence on the right, in order to borrow when there is need of it.

14. *Second Remark.* — It is clear that instead of diminishing by one unit the figure from which we have borrowed it, we can leave this figure unchanged, provided we augment the corresponding figure below by one unit. This manner of operating is in general more convenient in practice.

Thus, in the last example, after having said for the simple units, 7 from 11 leave 4, instead of saying for the tens, 8 from 9 leave 1, we say, 9 from 10 leave 1; in the same manner, instead of saying for the hundreds, 7 from 13 leave 6, we say, 8 from 14 leave 6, and so on for the rest.

But when we employ this modification, we must be careful to augment the lower figure only when we have been obliged to borrow in the subtraction of the preceding figures. This modification is used particularly in division.

VERIFICATION OF ADDITION AND SUBTRACTION.

15. We call *the verification of an arithmetical operation*, another operation which we perform in order to convince ourselves of the accuracy of the first.

The verification of addition is effected by adding anew, but commencing at the left hand. *After having formed the sum of the figures in the first column on the left, we subtract it from that part which answers to it in the sum total; we write down the remainder, which we reduce mentally into units of the order of the following figure, in order to join them to the units of this order in the sum total. In the same manner we sum up the second column on the left, and subtract this partial sum from the corresponding part of the sum total; we continue this operation to the last column; the last subtraction leaves no remainder.*

Thus, after having found that the four numbers,

5047
859
3507
846

have for their sum 10259

in order to verify the result 2120

we add the same numbers commencing on the left. We say, 5 and 3 make 8 thousands, which we subtract from 10 thousands, leaving 2 thousands for remainder; which, with the figure 2 hundreds, make 22 hundreds; then 8 and 5 make 13, and 8 make 21, which we take from 22, which gives for remainder 1 hundred, which, joined to 5 tens, forms 15 tens; 4 and 5 make 9, and 4 make 13; 13 from 15, there remains 2, which, joined to the 9 units following, gives us 29; lastly, 7 and 9, and 7 and 6 make 29; 29 from 29 and nothing remains; then the operation is exact.

The verification of subtraction is effected by adding to the smaller number the remainder found by the operation; and it is evident that we ought thus to reproduce the greater number, since this remainder is nothing more than the excess of the greater number over the less.

Thus, in the annexed examples, after having found that 54682 is the excess of the greater number over the less, if we add this excess to the number 28784, we ought to obtain the number 83466 — which we do in fact obtain.

$$\begin{array}{lr} & 83466 \\ & 28784 \\ \cline{2-2} \textit{Rem.} & 54682 \\ \cline{2-2} \textit{Proof} & 83466 \end{array}$$

16. Here we give some examples of addition and subtraction, with their *verifications*.

Additions.

$$\begin{array}{r} 83054 \\ 256870 \\ 748759 \\ 90874 \\ 130909 \\ 8746 \\ \hline 1319212 \\ \hline \cancel{3}\cancel{2}\cancel{4}\cancel{3}\cancel{3}\cancel{0} \end{array} \qquad\qquad \begin{array}{r} 700548 \\ 897597 \\ 6588 \\ 69764 \\ 407300 \\ 987846 \\ 1207047 \\ \hline 4276690 \\ \hline \cancel{3}\cancel{2}\cancel{4}\cancel{3}\cancel{3}\cancel{4}\cancel{0} \end{array}$$

Subtractions.

$$\begin{array}{r} 4073050062 \\ 2803767086 \\ \hline 1269282976 \\ \hline 4073050062 \end{array} \qquad\qquad \begin{array}{r} 20004001003 \\ 8405128605 \\ \hline 11598872398 \\ \hline 20004001003 \end{array}$$

Problem. — A banker had in his chest a sum of \$65,750*; he gave one person* \$13,259*; to a second,* \$18,704*; to a third,* \$22,050*; to a fourth,* \$9850*; what was the state of his chest after these payments?*

Solution. — After having summed up the four sums successively paid, we subtract the sum total from that which he had, and the result of the subtraction will be what ought to remain in his chest. Thus,

13259
18704
22050
9850
63863
21110

65750
63863
$1887 what he has left.

We remark, that in effecting the preceding addition and subtraction, we have considered the given numbers as abstract, although they were denominate numbers according to the enunciation of the question; but, arrived at the result, 1887, we have given it the name of the species of unit which the numbers expressed in the enunciation. We must always perform the operations in this manner, when we wish to apply the results of the operations to questions in denominate numbers. The results being altogether independent of the nature of the numbers, we consider them in a point of view purely abstract, except in giving to the final result the name of the unit which the enunciation of the question indicates.

MULTIPLICATION.

17. *To multiply one number by another, is to compound a third number with the first, as the second is compounded with unity.* Then, if the two given numbers are entire numbers, to multiply them is, *to take the first as many times as there are units in the second.*

We call the result of multiplication, *product;* the number to be multiplied, *multiplicand;* and the number by which we multiply, *multiplier;* which denotes how many times we are to take the first. The two numbers bear jointly the name of *factors of the product.* Properly speaking, multiplication is nothing else than addition; for, in order to obtain the result, it would suffice to write the multiplicand as many times as there are units in the multiplier, and then add all these numbers together. But this manner of operating would be very long, if the multiplier was composed of several figures; we are then to seek a method of simplifying it, and it is in this abbreviation that multiplication consists.

18. As long as the two factors are expressed, each one by a single figure, their product is obtained by the successive addition of the multiplicand to itself; thus, in order to multiply 7 by 5, we say, 7 and 7 make 14, and 7 make 21, and 7 make 28, and 7 make 35; this last number being the result of the addition of five numbers equal to 7, expresses the product of 7 by 5.

Beginners will do well to exercise themselves in this sort of multiplication; for they ought to impress the results upon the memory, if they wish subsequently to obtain easily the product of numbers expressed by several figures. Nevertheless, for those who are sufficiently exercised, all that is necessary is to give a table called *the multiplication table, or table of Pythagoras*, from the name of its inventor, or at least from him who first brought it into public use.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The first horizontal row of this table is formed by adding 1 to itself up to 9; the second, by adding 2 to itself; the third, by adding 3; and so on for the rest. We remark, moreover, that the same arrangement is made in the vertical columns. Each vertical column, taken in order, is composed of the same numbers as each horizontal row. Thus, the sixth horizontal row is composed of 6, 12, 18...54, and the sixth vertical column is composed of the same numbers, 6, 12, 18...54.

That being established, when we wish to obtain the product of two numbers from this table, we seek the multiplicand in the first horizontal row, and go down from this number vertically, until we arrive at that one which is opposite to the multiplier, which we find in the first vertical column. This number, contained in the little square, is the product. For example, in order to find the product of 8 by 5, we descend from 8, taken in the first horizontal row opposite to 5 in the first vertical column, and the number 40 in the little square is the required product.

19. Suppose, now, that *the multiplicand consists of several figures, and the multiplier of a single figure.*

OPERATION.

$$\begin{array}{r}8459\\8459\\8459\\8459\\8459\\8459\\8459\\\hline 59213\end{array}$$

Let it be proposed to multiply 8459 by 7. We could (17) obtain the result by writing one under another seven numbers equal to 8459, and adding successively the simple units, the tens, hundreds, &c., together. We would thus find 59213 for a result. But it is evident that this is nothing more than taking 7 times the 9 units of the multiplicand, 7 times the 5 tens, &c., and then to take the sum of all the products.

$$\begin{array}{r}8459\\7\\\hline 59213\end{array}$$

Thus, after having placed the multiplier, 7, under the multiplicand, we say, 7 times 9 make 63, (see table of multiplication), or 6 tens and 3 units; we place the 3 under the units, and reserve the 6 tens in order to add them to the product of the tens of the multiplicand by 7. We thus say, 7 times 5 make 35, and 6 make 41 tens, or 4 hundreds and 1 ten; we place 1 in the column of tens, and reserve the 4 hundreds; 7 times 4 make 28, and 4 make 32 hundreds, or 3 thousands and 2 hundreds; we place 2 in the column of hundreds, and retain the 3; lastly, 7 times 8 make 56, and 3 make 59; we write down the 9, and carry the 5 one place to the left, because there are no more figures in the multiplicand to be multiplied. We find thus, 59213 for the required product.

Whence we see that, *in order to multiply one number of several figures by another of a single figure, we must multiply successively the units, tens, hundreds, &c., of the multiplicand by the multiplier, and write these different partial products in the columns to which they belong, taking care at each partial multiplication, to reserve the tens in order to add them to the tens, the hundreds in order to add them to the hundreds, &c.*

OPERATION.

$$\begin{array}{r} 37008 \\ 9 \\ \hline 333072 \end{array}$$

Let it be proposed as a second example to multiply 37008 by 9. We say, first, 9 times 8 make 72; we write 2 in the column of units, and reserve the 7. Then 9 times 0 give 0; but we have reserved 7 from the preceding operation, so we write these 7 tens in the column of tens; 9 times 0 make 0; we write 0 in the rank of hundreds, since there are none, and since it is necessary to preserve the place of hundreds; then 9 times 7 make 63; we set down 3 and reserve 6; lastly, 9 times 3 make 27, and 6 make 33; we set down 3, and advance 3 one place to the left. Thus, the required product is 333072.

20. Before passing to the case in which the multiplier is composed of two or more figures, we will explain the method of rendering a number 10, 100, 1000 times greater, or of multiplying it by 10, 100, 1000.

It results from the fundamental principle of numeration (5), that if we place a 0 to the right of a number already written, each one of the significant figures of the number being thus advanced one step towards the left, expresses units ten times greater than before. In the same manner, by placing two 0's to the right, we render it 100 times as great, because each significant figure expresses units 100 times as great.

Then, *in order to multiply any entire number whatever by* 10, 100, 1000, *&c., it suffices to annex* 1, 2, 3,...*zeros.*

Thus, the products of 439 by 10, 100, 1000, 10,000, &c., are 4390, 43,900, 439,000.

21. *Let us consider now the case in which the multiplicand and multiplier are composed of several figures.*

OPERATION.

We propose to multiply 87468

By 5847

$$\begin{array}{r} 87468\\ 5847\\ \hline 612276\\ 3498720\\ 69974400\\ 437340000\\ \hline 511425396 \end{array}$$

We commence by placing the multiplier under the multiplicand, so that the units of the same order fall in the same column. This being arranged, we observe that, to multiply 87468 by 5847, is to take the multiplicand, 7 times, 40 times, 800 times, and 5000 times; then to add together these partial products. We can first find, by the rule of (19) the product of 87468 by 7, which gives 612276. But how obtain that of 87468 by 40? Let us conceive, for an instant, that we have written, one under another, 40 numbers equal to 87468, and that we make the addition of these numbers; we will thus have the required product. But it is evident that these 40 numbers form ten divisions, each division containing 4 times 87468. We form this product by rule (19), and find it to be 349872. Multiplying this product by 10, which (20) is effected by annexing a 0, we obtain 3498720 for the product of 87468 by 40.

We see, then, that this second operation reduces itself to multiplying the multiplicand by the figure 4, considered as expressing simple units, in writing a 0 to the right of the product, and in placing the result as we see above, below the first partial product. In like manner, in order to perform the multiplication of 87468 by 800, it suffices to multiply 87468 by 8, which gives 699744; then annex two 0's to the right of this product; we thus have a third partial product, 69974400, which we place below the two preceding products. For 800 numbers, equal to 87468, and, placed one under another, form evidently 100 divisions of 8 numbers, each equal to 87468, or 100 numbers, equal to the product of 87468 by 8; that is to say, 6997400 We could prove by a similar course of reasoning, that, in order to multiply by 5000, it suffices to multiply by 5, to annex three zeros to the product, and write the result, 437340000, thus obtained, below the three first products. Performing now the addition of these four partial products, we find at last the *total product*, 511425396.

N. B. — In practice, we dispense ordinarily with adding the zeros to the right of the partial products, found by multiplying by the figures in the tens, hundreds,...... places; but we write each partial product below the preceding product, advancing it one place to the right with reference to this product; that is to say, we make its last figure occupy the same column which the figure by which we multiply, occupies.

General Rule. — In order to multiply a number of several figures by a number of several figures—*Multiply first the multiplicand by the units figure of the multiplier, after the rule of* (19); *multiply in the same manner the whole multiplicand successively by the tens figure, by that of hundreds, &c., considered as simple units, and write the partial products one under the other, so that each one is advanced one column to the left, with reference to the preceding; then add these products; the result will be the total required product.*

22. Often some of the figures of the multiplier are zeros, and then it is necessary to make some modifications in the arrangement of the partial products.

OPERATION.

Multiply 870497

By 500407

We multiply, first, the whole multiplicand by 7, which gives for a product 6093479. Now, as there are no tens in the multiplier, we pass to the multiplication by 4, the hundreds figure, which gives the product, 3481988; and, since it is necessary to make it express hundreds, we place it under the first product, advancing it two columns to the left. In like man-

$$\begin{array}{r} 870497 \\ \underline{500407} \\ 6093479 \\ 3481988 \\ \underline{4352485} \\ 435602792279 \end{array}$$

ner, as there are no *thousands,* nor *tens* of *thousands* in the multiplier, we pass to the multiplication by 5, the figure in the place of *hundreds* of *thousands,* and write the product, 4352485, under the preceding, advancing it three places to the left, with reference to that one.

In general, *when there are one or more zeros between two significant figures of the multiplier, we advance the product corresponding to the significant figure, which is to the left of these zeros, one more column to the left than there are zeros between the figures.*

In fine, in order to avoid all error on this subject, we must take care at each operation that the last figure of each partial product falls *in the column of units of the same order as that of the figure by which we multiply.*

23. If one of the two factors of the multiplication, or both, are terminated by zeros, we abridge the operation by multiplying them as if the zeros were not there; but we place them at the end of the product.

EXAMPLE.

OPERATION.

Multiply 47000
By 2900

423
94
136300000

After having multiplied 47 by 29, according to the known method, we annex 5 zeros to the right of the product, and thus obtain 136300000 for the required product. For, if we had at first to multiply 47000 only by 29, it would be necessary to make the product express *thousands* (i. e.) units of the same species as the multiplicand; thus we ought to add 3 zeros. But to multiply a number by 2900, is (21) to take 100 times the product by 29; then we must add *two* new zeros. The same reasoning applies to all similar cases.

24. But little reflection on the method of multiplication will convince us of the necessity of commencing the operation on the

right, *at least in the partial multiplication by each one of the figures of the multiplier,* because of the *reservations* of figures which we frequently make in multiplying each figure of the multiplicand by each figure of the multiplier.

But nothing prevents us from inverting the order of the partial multiplications by the different figures of the multiplier, as we can see in the following example.

We have here commenced the multiplication with the *hundreds* figure of the multiplier; but in the following operation we have taken care to advance the product one column to the *right.* In the same manner the third product is advanced one place to the right with reference to the preceding. Usage alone requires us to form the products from right to left; it is also the more natural and convenient method.

OPERATION.

$$\begin{array}{r} 5704 \\ 487 \\ \hline 22816 \\ 45632 \\ 39928 \\ \hline 2777848 \end{array}$$

25. We will close the subject of multiplication by the explanation of several properties, of which we will often have to make use.

1st. *Let it be required to multiply* 345 *by* 72, *equal to* 8 *multiplied by* 9. We say, that to multiply 345 by 72, is to multiply 345 by 9, and the result by 8.

In order to establish this proposition without performing the operations, we must employ a mode of reasoning analogous to that employed in (21). To multiply 345 by 72, is to sum up 72 numbers equal to 345. But these 72 numbers, written one under another, form evidently 8 divisions of 9 numbers, equal to 345; then, after having multiplied 345 by 9, we must take this product 8 times. Thus, to multiply 345 by the product 72 of the two factors, 9 and 8, is to multiply 345 by 9, and the new result by 8.

As 9 is itself equal to the product of 3 and 3, we can say, that to multiply 345 by 72, is to multiply 345 by 3, the result obtained by 3, and finally the new result by 8. As we can apply this reasoning to other numbers, this general proposition results from it: *to multiply a number by a product of two or more numbers already formed, amounts to the same thing as multiplying the number by each one of the factors successively.*

26.—2d. In a multiplication of two factors, *we can take indifferently the first number for multiplicand, the second for multiplier, or reciprocally* In other terms—*the product of two numbers is the same in whatever order we perform the operation.*

Thus, the product of 459 by 237, is equal to the product of 237 by 459.

For, let us conceive unity written 459 times in a horizontal line, and let us form 237 of these lines; it is clear that the sum of the units contained in such a figure is equal to as many times the 459 units of a horizontal row as there are units in a vertical column, or in 237, (i. e.) that this sum is equal to the product of 459 by 237. But we can say also, that this sum is equal to as many times the 237 units of the vertical column as there are units in a horizontal row, or in 459; that is to say, is equal to the product of 237 by 459. Then, &c. If the nature of a question conducts to the multiplication of 76 by 5672, according to the proposition which we have just demonstrated, we would prefer to take the product of 5672 by 76, because in that case we would only have two partial products to form, while in the other operation we would have to form four of them. This proposition will be demonstrated for any number whatever of factors.

1, 1, 1, 1, 1, ...
1, 1, 1, 1, 1, ...
1, 1, 1, 1, 1, ...
1, 1, 1, 1, 1, ...
. . . .
. . . .

DIVISION.

27. To divide one number by another, is *to find a third number, which, multiplied by the second, will reproduce the first;* or, in other terms, being *given the product and one of the factors, to determine the other factor.* As in the multiplication of entire numbers, the product is composed of as many times the multiplicand as there are units in the multiplier, we can also say, that, *to divide one entire number by another, is to seek how many times the first number, considered as a product, contains the se-*

cond, considered as multiplicand; the number of times is then the multiplier. Finally, we can also say, that, *to divide a number by another, is to divide the first number into as many equal parts as there are units in the second.*

These last two points of view, under which we sometimes consider division, pertain only to entire numbers, while the two first pertain to all possible numbers, whether entire or fractional. Nevertheless, the names given to the terms of division have been drawn from these last two points of view.

Thus, the first number is called *dividend*, the second is called *divisor*, and the third *quotient*, from the Latin word *quoties;* because it expresses how many times the dividend contains the divisor.

It results, obviously, from the first two definitions, that when we have obtained the quotient, in order to make the verification of the operation, it will suffice to multiply the divisor by the quotient; and, if the operation has been exact, we will thus reproduce the dividend.

Reciprocally in multiplication, the product may be considered as the dividend, the multiplicand as the divisor, and the multiplier as the quotient; thus, we make the verification of multiplication by dividing the product by one of the factors; and if the operation is exact, we ought to reproduce the other factor. These ideas being established, we pass to the explanation of the method of division.

28. In the same manner as multiplication can be effected by the addition of a number several times to itself, we can also find the quotient of a division by a series of subtractions.

For, let it be required to divide 60 by 12. As many times as we can subtract 12 from 60, so many times is 12 contained in 60. Thus, the quotient is equal to the number of subtractions which we can make before the dividend is exhausted.

```
           60
           12
1st rem.   48
           12
2d rem.    36
           12
3d rem.    24
           12
4th rem.   12
           12
5th rem.    0
```

In this example, as we are obliged to make 5 subtractions, it follows that the quotient is 5. But this manner of performing the division would be too long in practice, especially if the dividend was very great in comparison with the divisor. It is in the art of abridging the operation that the ordinary method of division consists.

29. From the fact that we know by heart the products of two numbers of a single figure, we can determine easily the quotient of the division of a number of one or two figures by a number of a single figure.

For example, 35 divided by 7, gives for a quotient 5. This we know, because we know that 7 times 5 give 35. We say, also, in this example, that the 7th of 35 is 5, because 7 times 5 make 35. Suppose, again, that we have to divide 68 by 9. As 7 times 9, or 63, and 8 times 9, or 72, comprise 68 between them, it follows that 68, divided by 9, gives for the quotient, 7, with a remainder, 5; or the 9th of 68 is 7, with a remainder, 5.

In like manner, 47 contains 8, 5 times, with a remainder 7; because 5 times 8 gives 40, and 6 times 8 gives 48.

We will see farther on what is to be done with the remainder, when the divisor is not contained exactly in the dividend.

30. Let us consider the case in which the dividend is composed of any number of figures, the divisor containing but a single figure.

Divide 6766453 by 8.

```
6766453)8
64       845806
 36
 32
  46
  40
   64
   64
   053
    48
     5
```

Proof by multiplication.

```
845806
     8
6766448
      5
6766453
```

After having written the divisor to the right of the dividend, and separated them by a vertical line, we draw below the divisor a horizontal line. This arranged, we see at once that, if we place (mentally) to the right of the divisor, 8, five zeros, (i. e.) multiply it by 10,000, then six zeros, or multiply it by 100,000, the two products, 80,000 and 800,000, are the one smaller, the other greater than the dividend. Whence we conclude that the quotient demanded is comprised between 10,000 and 100,000; that is to say, is composed of six figures, and that thus the highest units of the quotient are hundreds of thousands, of which we must find the figure.

Now, as the product of the divisor by the figure sought cannot give units of a lower order than hundreds of thousands, it follows, that this product is contained wholly in the 67 hundreds of thousands of the dividend; and if we divide 67 by 8, which gives the quotient 8 for 64, and the remainder 3, we can affirm that the figure of hundreds of thousands in the quotient is 8. In fact, 800,000 times 8 gives 6,400,000, a number which can be subtracted from the dividend, 6766453; while 900,000 times 8, or 7,200,000 cannot be so subtracted. The figure 8 being thus determined, we place it under the divisor; then we subtract the product 8 by 8, or 64 from 67, and conceive the remaining figures of the dividend to be written to the right of the remainder 3, which gives 366453 for the total remainder of this first operation. (This first operation is evidently nothing more than subtracting from the dividend 800,000 times the divisor, or is equivalent to 800,000 successive subtractions of the divisor 8.) It would seem necessary to write on the right of the quotient already obtained, five zeros, in order to give it its true value; but we avoid this by the arrangement which we will make of the following figures of the quotient.

We must now determine the figure of tens of thousands of the quotient. Since the product of the divisor by this figure cannot give units of an order inferior to tens of thousands, it is contained wholly in the 36 tens of thousands of the remaining dividend. It suffices then to bring down to the side of the remainder, 3, the following figure, 6, of the dividend; then to divide 36 by 8, which gives the quotient, 4, for 32, and the remainder, 4. We write this quotient, which expresses necessarily the tens of thousands of the whole quotient, on the right of the first quotient, 8; then, after having subtracted 4 times 8, or 32 from 36, we bring down to the right of the 4, the next figure of the dividend, which gives 64. (This new operation, which amounts to subtracting 40,000 times 8, or 320,000 from 366,453, is equivalent to 40,000 new successive subtractions of the divisor, 8.)

In order to obtain the ones of thousands of the whole quotient, we divide 46 by 8; the quotient is 5 for 40, and the remainder, 6. We write this new quotient, 5, to the right of the first two; then, after having subtracted 5 times 8, or 40, from 46, we bring down to the right of the remainder, 6, the next figure, 4, of the dividend, which gives 64. (This third operation is equivalent to 5000 successive subtractions of the divisor, 8.)

In order to obtain the figure of hundreds of the total quotient, we divide 64 by 8, which gives 8, and 0, for remainder; we write the new quotient to the right of the three first; then, after having subtracted 8 times 8, or 64 from 64, we bring down to the right of the remainder, 0, the next figure of the dividend, which gives 05, or simply 5.

Here a particular case presents itself; as the new partial dividend, 05 or 5, which is to give the tens of the quotient, is less than the divisor, 8, we must conclude that the total quotient has no tens, (and in fact the remaining dividend is 53, a number less than 10 times 8, or 80.)

We place, then, a 0 in the quotient to the right of the four figures already obtained, in order to replace the tens which are wanting, and preserve the relative value of the preceding and following figures; we then bring down to the right of the remainder, 5, the next and last figure of the dividend, and continue the operation. The quotient of 53 divided by 8, being 6 for 48, we write this figure to the right of the first five quotients already found; we then subtract 48 from 53, which gives at last 5 for the remainder of the entire operation; and the required quotient is 845806, which we can easily verify by multiplying 8 by 845806, or rather 848806 by 8, and adding the remainder, 5, to the product thus obtained. (All the operations which have been performed in effecting this division are equivalent, evidently, to 800,000, then 40,000 subtractions, then 5000, then 800, then 6, or 845806 successive subtractions, in which the divisor, 8, is constantly the number to be subtracted.)

31. We will not establish for the case of division which we have just discussed, a general rule founded on the preceding reasoning, because there exists (for this case only) a practical method, more convenient and more simple in reference to the arrangement of the calculations. Let us take again the above example:

6766453 to be divided by 8.
Quotient, 845806; remainder, 5.

We know already (No. 27) that to divide a number by 8, or to seek how many times 8 is contained in this number, amounts to dividing the number into 8 equal parts, or taking the eighth of it. This being fixed, taking the two first figures to the left of the dividend, 67, we say, the eighth of 67 is 8, with the remainder, 3. We write the quotient, 8, under the figure, 7, of the dividend; then we place, mentally, the remainder, 3, expressing 3 hundreds of thousands, or 30 tens of thousands, to the left of the figure, 6, of the dividend, which expresses also tens of thousands; we say, as before, the eighth of 36 is 4, with remainder 4. We write the second quotient, 4, to the right of the first; placing again, mentally, the remainder, 4, expressing 4 tens of thousands, or 40 thousands, to the left of the thousands figure, 6, of the dividend; we say, again, the eighth of 46 is 5, with the remainder, 6; we write the third quotient, 5, to the right of the preceding; continuing in the same manner, we say, again, the eighth of 64 is 8, with the remainder, 0, and we write the fourth quotient, 8, to the right of the third. The eighth of 05, or 5, is 0, with the remainder, 5; we write this fifth quotient to the right of the fourth. Finally, the eighth of 53 is 6, with the remainder, 5; we write to the right of the fifth quotient the sixth and last partial quotient, which thus falls beneath the units figure of the dividend, and we have for the result the quotient, 845806, with the remainder, 5.

Second example:

8230200409 to be divided by 6.
Quotient, 1371700068; remainder, 1.

Here, the first figure on the left of the dividend being greater than the divisor, we see that the quotient ought to have units of the same order as those of the figure 8; and we say, the sixth of 8 is 1, which we write under the figure 8, with the remainder, 2; then the sixth of 22 is 3, which we place to the right of the figure 1, with the remainder, 4.

The 6th of 43 is 7, with the remainder, 1.
The 6th of 10 is 1, with the remainder, 1.
The 6th of 42 is 7, with the remainder, 0.
The 6th of 0 is 0, with the remainder, 0.
The 6th of 0 is 0, with the remainder, 0.
The 6th of 4 is 0, with the remainder, 4.
The 6th of 40 is 6, with the remainder, 4.
Finally, the 6th of 49 is 8, with the remainder, 1.

The required quotient is then 1371700068, with the remainder, 1.

It is very important to understand thoroughly this method, because it finds its application in the case of division, which is yet to be discussed.

We will observe, moreover, that when we know by heart the multiplication table as far as the number 12, we can obtain very easily, by the same method, the 10th, 11th, and 12th, of any number whatever.

EXAMPLES.

1st. 897614708497, to be divided by 12.
Quotient, 74801225708, remainder, 1.

(The 12th of 89 is 7, with remainder, 5; the 12th of 57 is 4, with remainder, 9; the 12th of 96 is 8, remainder, 0; &c., &c.)

2d. 23054273896, to be divided by 11.

Quotient, 2095843081; remainder, 5.

(The 11th of 23 is 2, with remainder, 1; the 11th of 10 is 0, with remainder, 10; the 11th of 105 is 9, with remainder, 6; &c., &c.)

As to the division by 10, instead of applying the method, it is simpler to separate in thought the last figure to the right of the dividend. The part to the left expresses the quotient, and this last figure separated (which can be 0), is the remainder of the division. This is an evident consequence of the system of numeration.

Thus, the 10th of 2710548 is 271054, and the remainder, 8; the 10th of 863005704 is 86300507, and remainder, 4; the 10th of 3805670 is exactly 380567; results which can be found also by the application of the method above.

32. Let us pass to the case *in which the given numbers being both composed of several figures, the quotient is to have one only.*

This case deserves, of itself, particular attention; and it will serve us, besides, as a basis for the development of the general case.

Let it be given to divide 730465 by 87467.

730465	87467
699736	8
30729	

We remark, first, that the product of the divisor by 10, or 874670, is greater than the dividend; thus, the quotient sought is less than 10, and can have only one figure.

In the second place, the product of 8 tens of thousands of the divisor by the figure sought, as it cannot give units of an order inferior to tens of thousands, must be found wholly in the 73 tens of thousands of the dividend; whence it follows, that the figure sought cannot exceed the quotient of the division of 73 by 8. We are then conducted to the division of the part, 73, on the left of the dividend, by the first figure, 8, of the divisor, which gives the quotient, 9. But 9 is evidently too large; for, in the multiplication of the whole divisor by this figure, we find, in multiplying the thousands figure, 7, of the divisor, by 9, 63 units of this order, and, consequently, 6 tens of thousands, to be added to the 72 tens of thousands, product of the first figure, 8, of the divisor, by the same figure, 9; which would give 78 tens of thousands, a number greater than the dividend

It is not necessary, then, to try any figure higher than 8, as figure of the quotient required. Effecting the multiplication of 87467, by 8 (which we have placed under the divisor), we obtain a product of 699736, less than the dividend; which proves that the quotient, 8, is correct. On subtracting this product from the dividend, as the operation shows, we find for remainder, 30729.

Again, divide 974065 by 189768.

$$\begin{array}{r|l} 974065 & \underline{189768} \\ \underline{948840} & 5 \\ 25225 & \end{array}$$

As the dividend and the divisor are composed of the same number of figures, it is clear that the quotient ought to have only one figure; and in order to find it, we divide, first, the first figure on the left of the dividend, 9, by the first figure, 1, on the left of the divisor. The quotient is 9; but this figure, and the next lower, 8, 7, 6, are too large, if we consider the two first figures, 18, on the left of the divisor; for the products of 18, by 9, 8, 7, 6, being 162, 144, 126, and 108, all surpass the 97 tens of thousands of the dividend. This leads us to try the figure 5.

On multiplying the divisor by 5, we have the product, 948840, which, subtracted from the dividend, gives for remainder, 25225, a number smaller than the divisor; which proves that the quotient, 5, is not too small.

33. In the two preceding examples, we have been able to determine pretty easily what was the true figure of the quotient; but as this is not always the case, it is important to have a me-

thod of ascertaining, without effecting the product of the divisor by the quotient, whether the trial figure is the true one. We will now develop this method.

Particular method of trial.

Given, 556428, to be divided by 69784.

$$\begin{array}{r|l} 556428 & 69784 \\ \cline{2-2} 488488 & 7 \\ \cline{1-1} 67940 & \end{array}$$

The division of 55 (the two first figures on the left of the dividend), by 6 (the first figure on the left of the divisor), gives 9 for quotient, with the remainder, 1.

In order that 9 may not be larger than the quotient sought, 9 times the divisor must be less, or, at most, equal to the dividend; or, which is the same thing, the 9th of the dividend must be greater, or at least equal to the divisor.

We then commence to take the 9th of 556428, after the method of (31). We find for the two first figures on the left, 61 tens of thousands, a number less than 69 tens of thousands of the divisor, which shows that the 9th of the dividend is less than the divisor; 9 ought then to be rejected. We next try 8. We find for the three first figures of the 8th of the dividend, 695 hundreds less than the 697 hundreds of the divisor; then 8 is too large. We now try 7. The first figure of the 7th of the dividend is 7, greater than 6, the first figure of the divisor. Whence it follows, that the 7th of the dividend is greater than the divisor; or, in other terms, that the product of the divisor by 7, is less than the dividend. Thus, the figure 7 is the true one. Multiplying the divisor by 7, and writing the product, 488488, below the dividend, then effecting the subtraction, we obtain the remainder, 67940, a number smaller than the divisor.

Another Example.

Given, to divide 1148367 by 169987.

$$\begin{array}{r|l} 1148367 & 169987 \\ \cline{2-2} 1019922 & 6 \\ \cline{1-1} 128445 & \end{array}$$

The division of 11 by 1, would give 11 for a quotient; but the required quotient cannot be greater than 9, since the divisor, multiplied by 10, would be a larger number than the dividend.

Let us try 9; the 9th of the dividend is 12 . . . . smaller than the divisor, 16. . . . . We therefore reject the 9.

The 8th of the dividend is 14 . . . . less than 16 . . . . We therefore reject 8.

The 7th of the dividend is 164 . . ... less than 169 . . . . of the divisor. We then reject 7.

The 6th of the dividend is 19 . . . . greater than the divisor, 16; thus, the figure 6 is the true one. Multiplying the divisor by 6, and subtracting the product, 1019922, from the dividend, we obtain the remainder, 128445, smaller than the divisor.

The course to be pursued in this method of trial is evident from the exposition of the last example. We stop as soon as we obtain a figure greater or less than the corresponding figure of the divisor. If it is greater, we can affirm that the trial figure is the true one; if less, we know that the trial figure must be diminished.

We add, that all these trials can be made mentally, without writing anything.

It could happen (but rarely), that we reproduced thus successively all the figures of the divisor, in arriving at a remainder necessarily less than the trial figure, and possibly 0. We would then have not only the required quotient, but also the remainder of the proposed division, which would be nothing else than this final remainder.

We recommend especially the exercise of this method of trial, as a means of avoiding all difficulty in any case whatever.

34. *Second Remark.* — We have seen in all which precedes, that after having determined a figure of the quotient, we proceed to multiply the divisor by this figure, to write the product below the dividend, and to effect the subtraction, placing the remainder under this product. But we can employ a method of abbreviation which we will now explain.

Let us take the first example discussed in (32.)

$$\begin{array}{r|l} 730465 & 87467 \\ \hline 30729 & 8 \end{array}$$

This method of abbreviation consists in forming the product of the divisor by the figure 8 of the quotient, mentally, and in writing only the remainder below the dividend. To accomplish this, we must subtract, successively, from the units, tens, hundreds, &c., of the dividend, the products of the units of the same order of the divisor by the quotient, as fast as we form them mentally. Thus, in the example above, we have at first to subtract from the 5 units of the dividend the product, 56, of the quotient, 8, by the 7 units of the divisor; but as this subtraction is impossible (which will generally be the case), we add mentally 6 tens to the 5 units, with the express reservation of adding, by way of compensation, these 6 tens also to the product of the quotient, 8, by the tens of the divisor (to be subtracted in its turn); we form thus the number 65, from which we subtract 56, which gives for a remainder, 9, which we write below the units of the dividend. Passing to the tens figure of the divisor, 6, we say, 8 times 6 give 48 tens, which, augmented by 6 tens (reserved in the last operation), make 54 tens to be subtracted from the 6 tens of the dividend. In order to perform the subtraction, we add 5 hundreds (tens of tens), to these 6 tens, making 56 tens, from which we subtract the 54 tens, and write the remainder, 2, under the tens of the dividend. Continuing thus, we say, 8 times 4 hundreds make 32, and 5 (which were added in the preceding operation), make 37. This cannot be subtracted from 4; but 37 from 44 leave 7, which we write under the hundreds of the dividend.

In the same manner, 8 times 7 make 56, and 4 (added in the preceding), make 60; 60 from 60 leave 0, which we write under the thousands of the dividend. Finally, 8 times 8 make 64, and 6 make 70; 70 from 73, leave 3. The total remainder is then 30729.

Let us take, again, the second example. We abridge the discussion, making use of the expressions commonly used in practice (though often incorrect).

$$\begin{array}{r|l} 974065 & 189768 \\ \hline 25225 & 5 \end{array}$$

Having found 5 the true figure of the quotient, we say, 5 times 8 make 40; 40 from 45, leave 5, and carry 4 (understood, in order to add them to the following product); 5 times 6 make 30, and 4 make 34; 34 from 36, 2 remain, and carry 3; 5 times 7 make 35, and 3 make 38; 38 from 40, remain 2, and carry 4; 5 times 9 are 45, and 4 make 49; 49 from 54 leave 5, and carry 5. Finally, 5 times 18 make 90, and 5 make 95; 95 from 97 leave 2. The total remainder is, 25225.

N. B. It is very important, at each partial operation, to say, carry such a figure, in order not to forget the number which, by compensation, is to be added to the following product.

35. We have thus very fully discussed the simple cases of division, because, having once mastered these thoroughly, also the methods belonging thereto, of abbreviation and trial, the pupil will find no difficulty in comprehending the general case which we will now discuss, namely, *where the dividend, the divisor, and the quotient, contain any number of figures.*

GENERAL CASE OF DIVISION.

Given, to divide 9176298 by 2678.

$$\begin{array}{r|l} 9176298 & 2678 \\ \hline 11422 & 3426 \\ \hline 7109 & \\ \hline 17538 & \\ \hline 1470 & \end{array}$$

Verification by multiplication.

$$\begin{array}{r} 2678 \\ 3426 \\ \hline 16068 \\ 5356 \\ 10712 \\ 8034 \\ 1470 \\ \hline 9176298 \end{array}$$

We arrange here the parts of the division; the quotient, the successive remainders, according to the indications which precede; then we reason as in (No. 30).

If we place mentally three zeros, and afterwards four zeros to the right of the divisor, we obtain two products, 2678000, and 26780000, the one less, the other greater than the dividend. Thus, the total quotient is comprised between 1000 and 10,000, or must be composed of four figures, of which the first to the left expresses *thousands*. In order to find this first figure, we observe that its product by the divisor, inasmuch as it is thousands, is to be found wholly in the part, 9176 thousands of the dividend. We are then led to divide 9176 (which we consider as a first partial dividend), by 2678; and the greatest number of times that the second number is contained in the first, represents the *thousands* figure of the total quotient. Now, the true quotient of 9176, by 2678, obtained according to the method of trial indicated in (33), is 3. We write, then, 3 below the divisor; we next subtract from the dividend, the product of the divisor, by 3, either by placing this product below the partial dividend, and subtracting, or (No. 34), effecting simultaneously the subtraction and the multiplication, as the table above indicates. (This first operation amounts, evidently, to subtracting 3000 times the divisor from the dividend.)

The remainder of this first subtraction being 1142, if we write after it the figures of the dividend, which have not yet been used, there would result a new dividend, upon which we could operate as upon the primitive dividend; but, as we have now to determine the hundreds figure of the quotient, and as the product of the divisor by this figure cannot give units of a lower order than hundreds, it must be contained wholly in the 11422 hundreds of the remaining dividend; so we bring down to the right of the remainder, 1142, only the following figure, 2, of the dividend; which gives a second partial dividend, 11422, upon which we operate as on the first.

The true quotient of the division of 11422, by 2678, is 4, which we write below the divisor, and to the right of the first quotient obtained. We then subtract from the second partial dividend, the product of the divisor, by the new quotient. The remainder of this subtraction is 710. We bring down to its right the following figure of the dividend, 9, which gives a third dividend, 7109, which is to furnish the tens figure of the total quotient.

Dividing 7109 by 2678, we have for a true quotient, 2, which we write to the right of the two first figures of the quotient; multiplying the divisor by 2, and subtracting the product from the third partial dividend, we obtain 1753 for a remainder, to the right of which we bring down the last figure, 8, of the dividend, which gives 17538 for a fourth partial dividend. Finally, the true quotient of 17538 by 2678, is 6. We multiply the divisor by 6, and subtract the product from the fourth partial dividend, which gives a remainder, 1470. The required quotient is then 3426, with the remainder, 1470; which we can verify by multiplying 2678 by 3426, and adding 1470 to the product, as the table of operations shows. (The four operations which we have just performed in this division, conduct to the same result as if we had subtracted successively from the dividend, 3000 times, then 400 times, then 20 times, then 6 times the proposed divisor.)

Second Example.

Given, to divide 42206581591, by 569874.

42206581591	569874
2315401	74063
3590559	
1713151	
3529	

Placing mentally four zeros, then, five zeros, to the right of the divisor, we obtain two products, 5698740000, and 56987400000, which contain the dividend between them; which proves that the quotient sought is itself comprised between 10,000 and 100,000, or is composed of 5 figures, of which the first to the left expresses the tens of thousands. The two first figures on the left of this quotient, 74, which we have placed below the divisor, are found without difficulty, as in the first example. But, having arrived at the remainder, 35,905, if, in order to form the third partial dividend, which is to furnish the hundreds figure of the total quotient, we bring down to the right of this remainder the next figure, 5, of the dividend, we obtain 359055, a number less than the divisor, which proves that the quotient has no hundreds. We must then place a 0 to the right of the two first quotients, then bring down the next figure, 9, to the right of 359055; we thus have a fourth partial dividend, 3590559, which we divide by 569874, in order to obtain the tens of the quotient. Continuing the operation, we find, finally, the total quotient, 74063, with the remainder, 3529.

GENERAL RULE.

36. *In order to divide any two entire numbers whatever, one by another, write the divisor to the right of the dividend; separate them by a vertical, then draw a horizontal line below the divisor. This done, take on the left of the dividend the number of figures necessary and sufficient to contain the divisor; you obtain thus a first partial dividend, composed either of as many figures, or one more, than there are in the divisor. See how many times this partial dividend contains the divisor, and write the resulting quotient under the divisor; multiply the divisor by this figure, and subtract the product from the first partial dividend.*

Bring down to the right of the remainder, the next figure of the dividend, which gives a second partial dividend. See in the same manner how many times this second partial dividend contains the divisor, and write this new quotient to the right of the first. Multiply the divisor by this second quotient, and subtract the product from the second partial dividend.

Bring down to the right of this second remainder, the next figure of the dividend, which gives a third partial dividend, upon which you operate as upon the preceding.

Continue this series of operations, until you have brought down the last figure of the dividend, taking care at each operation to *write the quotient which you obtain to the right of the preceding, (in order to give to the former their proper value).*

If it happens after having brought down a figure, that you obtain a partial dividend less than the divisor, place a 0 *in the quotient, and then bring down a new figure to form a new partial dividend. When, after all these operations, we arrive at a remainder,* 0, *the dividend is said to be exactly divisible by the divisor; if the remainder is not* 0, *we add it in the verification to the product of the divisor by the quotient.*

37. From the nature of the method, we deduce the following consequences:

1st. No partial division can give a quotient greater than 9, nor ought to lead to a remainder less than the divisor.

2d. The first figure on the left of the quotient expresses units of the same order as those expressed by the first figure on the right of the first partial dividend; and consequently the quotient contains one more figure than the rest of the dividend, after the first partial dividend has been separated. In other terms, *the number of figures of the quotient is either the difference between the number of figures in the dividend and the number of figures in the divisor, or this difference augmented by unity.*

Particular Cases of Division.

Remark.—When one of the terms of a division to be performed, or both of them, are terminated by zeros, we can simplify the general method.

We will examine, specially, the case in which the divisor alone is terminated by zeros, as the same rule of simplification can be applied to all other cases, a single one excepted.

1st. Given, 47543296 to be divided by 690000.

General method.		*Particular method.*		
47543296	690000	4754	3296	69
6143296	68	614		68
623296		*Rem.* 62		*True rem.* 623296

The rule to be followed is,—suppress the zeros which terminate the divisor, and separate on the right of the dividend as many figures as there are zeros on the right of the divisor. There remains then to be divided, 4754 by 69; perform this division after the ordinary method. The quotient obtained, 68, is the true quotient of the division of the given numbers. Write after the corresponding remainder, 62, the figures 3296 of the dividend which have been separated to the right, and you have 623296 for the total remainder of the division.

This manner of operating rests on the following:—We observe, first, that the 69 tens of thousands of the primitive divisor are contained in the 4754 tens of thousands of the dividend, the same number of times that 69 simple units are contained in 4754 simple units. Thus, the quotient of 4754 by 69 ought to be identical with the quotient of the division of the two given numbers.

In the second place, the remainder of the division of 4754 by 69, being less than the divisor, 69, it follows that this remainder, followed by the figures to the right of the dividend, 3296, is less also than the divisor followed by four zeros, or 690000; then, 623296 expresses the true remainder of the division of the two numbers, 47543296 and 690000, (as 620000 would be the remainder, if the dividend were 47540000. So 623296 must be the true remainder in the case above).

2d. The case in which the dividend alone is terminated by zeros, does not give place for any simplification. Nevertheless, if the part to the left of these zeros is to contain the divisor exactly, we could first cut off the zeros; then, after having obtained the quotient of the division of the part on the left by the divisor, we would write to the right of this quotient the zeros which terminate the dividend.

EXAMPLE.

Given, 375000 to be divided by 125. The division of 375 by 125, giving 3 for exact quotient, the quotient demanded is 3, followed by three zeros of the dividend, or 3000. But this simplification is of no importance.

3d. The same number of zeros may be on the right of both dividend and divisor. In this case, we suppress the zeros in the two terms of the division; then, after having divided the two parts on the left, one by the other, we write after the remainder thus obtained, the zeros on the right of the dividend.

Given, to divide 5679800 by 8600.

$$\begin{array}{r|l} 56798\vert 00 & 86\vert 00 \\ \hline 519 & 660 \\ \text{Rem. } 38 & \end{array} \qquad \textit{True rem. } 3800$$

This mode of operating is a repetition of the first case, with the sole difference that the remainder of the division of 56798 by 86, or 38, is to be followed by the zeros which terminate the primitive dividend, instead of being followed by significant figures.

4th. Fewer zeros on the right of the dividend than on the right of the divisor.

Given, to divide 68235947000 by 547600000.

$$\begin{array}{r|l} 682359\vert 47000 & 5476 \\ \hline 13475 & 124 \\ 25239 & \\ \text{Rem. } 3335 & \end{array} \qquad \textit{True rem. } 333547000$$

This case is compounded of the first and third.

5th. More zeros on the right of the dividend than on the right of the divisor.

Given, to divide 25036900000 by 875000.

$$\begin{array}{r|l} 25036900\vert 000 & 875 \\ \hline 7536 & 28613 \\ 5369 & \\ 1190 & \\ 3150 & \\ \text{Rem. } 525 & \end{array} \qquad \textit{True rem. } 525000$$

This case is, properly speaking, nothing more than a particular case of the first case.

General Remark. — As in the three first operations of arithmetic, the calculations are performed by commencing on the right, it is natural to ask why, in division, we commence on the contrary on the left.

In order to answer this question, we must observe that the dividend, being the sum of the partial products of the divisor, by the units, tens, hundreds, &c., of the quotient, all these partial products are mingled one with another; so that it is impossible to commence by separating out the products by the units, by the tens, &c.; while, by the established method we determine at once in what part of the dividend the product by the highest units is found, and then we obtain the figure of these highest units; then we arrive at the figure of the units of the order immediately below the first, and thus with the rest.

Two Examples for Exercise.

1st. 12187610837 to be divided by 15619.
Quotient, 780306; remainder, 11423.

2d. 2487623393304 to be divided by 5076078.
Quotient, 490068; remainder, 0.

Verification of Multiplication and of Division.

It has been established in (No. 27), that we are naturally led by the definition, even of division, to make the proof of *Multiplication* by *Division*, and that of *Division* by *Multiplication;* and we have given in the course of the exposition of the method, the means of performing this operation. But we will show later more expeditious methods of making these verifications.

38. We will now give some uses of multiplication and division.

Question 1st. — Required the price of 2564 yards of a piece of masonry, of which each yard costs 47 dollars.

Since each yard costs 47 dollars, it is clear that by repeating this value 2564 times, we will have the price of the 2564 yards. Thus, it suffices to form the product of 47 by 2564, or rather the product of 2564 by 47; and this product will express the number of dollars required.

We give the operation and its proof by division.

$$\begin{array}{r} 2564 \\ 47 \\ \hline 17948 \\ 10256 \\ \hline 120508 \end{array} \qquad\qquad \begin{array}{l|l} 120508 & 47 \\ \cline{2-2} \underline{265} & 2564 \\ \underline{300} & \\ \underline{188} & \\ 000 & \end{array}$$

The 2564 yards cost $120,508.

Question 2d.—One yard of a piece of masonry cost $39. Required how many yards can be built for $8395.

It is evident that as many times as 39 will be contained in 8395, so many yards can be constructed for the price. Thus, it suffices to divide 8395 by 39; and the quotient will be the required number of yards.

$$\begin{array}{l|l} 8395 & 39 \\ \cline{2-2} \underline{59} & 215\frac{10}{39} \\ \underline{205} & \\ 10 & \end{array} \qquad\qquad \textit{Proof.} \quad \begin{array}{r} 215 \\ 39 \\ \hline 1935 \\ 645 \\ 10 \\ \hline 8395 \end{array}$$

As we obtain 215 for quotient, and 10 for remainder, it is necessary to know the use which we are to make of this remainder.

Let us observe, that if the dividend contained $10 less, it would be the product of 39 by 215. Thus, the number of yards demanded would be 215; but, as we have 10 dollars more, we have to determine the part or fraction of a yard which can be constructed for these 10 dollars.

Now, with one dollar, we would evidently have $\frac{1}{39}$th of a yard, since we could build the whole yard for $39; then, with $10, we must have 10 times $\frac{1}{39}$, or (No. 8), $\frac{10}{39}$. Thus, 215 yards, and $\frac{10}{39}$ths of a yard, is the result required.

Such is, in general, the use which we have to make of the remainder of a division, when in performing the operation we are resolving a question relating to concrete numbers.

We conceive the unit of the quotient (the nature of which is always determined by the enunciation of the question), *to be divided into as many equal parts as there are units in the divisor; we take one of these parts as many times as there are units in the remainder; we then add the resulting fraction to the entire quotient already obtained.*

Question 3d.—Suppose that 498 persons have to divide equally a sum of \$1,348,708. Required the part of each one.

$$\begin{array}{r|l} 1348708 & 498 \\ \cline{2-2} 3527 & 2708\frac{124}{498} \\ 4108 & \\ 124 & \end{array} \qquad\qquad \begin{array}{r} 2708 \\ 498 \\ \hline 21664 \\ 24372 \\ 10832 \\ 124 \\ \hline 1348708 \end{array}$$

The quotient of this division being 2708, and the remainder 124, we conclude that if the sum to be divided was diminished by 124 dollars, each person would have for his portion 2708 dollars. But, as the given sum contains \$124 more than the product of 2708 by 498, it follows that each person ought to have \$2708, and a part of \$124. In order to form an idea of this part, we can at first consider the number 124 as a whole, which it is necessary to divide into 498 equal parts; and one of these parts is the fraction which is to complete the quotient: but it is simpler to conceive the unit (No. 8), 1 dollar, to be divided into 498 equal parts, and to take 124 of these parts, which gives $\frac{124}{498}$ for the fraction to be added to the entire quotient.

39. N. B. — This last example leads us to a remark of which we will often make use; it is, that to divide a number, 124, into 498 equal parts, is to take 124 times the 498th part of unity. For, if instead of 124, we had simply to divide 1 into 498 equal parts, each part would be (No. 8), $\frac{1}{498}$; but, as the number to be divided is 124 times greater, the result of the division ought to be 124 times greater, or equal to 124 times $\frac{1}{498}$, or equal to

$\frac{124}{498}$. *In general, to divide a number into as many equal parts as there are units in another, is the same thing as to divide unity into as many equal parts as there are units in the second number, and to take one of these parts as many times as there are units in the first.*

40. From the two propositions demonstrated in Nos. 25 and 26, we deduce some consequences which it is well to make known, as they are of continual use in arithmetic.

We observe, first of all, that according to the definitions, even of the multiplication and division of entire numbers, we render an entire number as many times greater or smaller as there are units in another, in multiplying or dividing the first number by the second.

Thus, when we multiply 24 by 6, the product which we obtain is 6 times as great as 24, since it results from the addition of 6 numbers equal to 24. In the same manner, if we divide 24 by 6, the quotient is 6 times as small as 24, since this quotient, repeated 6 times, reproduces 24.

This established, we say, first, that if in a multiplication we render the multiplicand or multiplier a certain number of times greater or smaller, the product is, by this change, rendered the same number of times greater or smaller.

Given, for example, to multiply 47 by 6, and suppose that, instead of performing this operation, we multiply 47 by 24, which is 4 times as great as 6; since, according to what has been said in (25), to multiply 47 by 24, is the same as to multiply 47 by 6, and the product by 4, it follows at once that the product of 47 by 24, equals 4 times the product of 47 by 6; (*i. e.*) is 4 times as great.

Reciprocally, the product of 47 by 6 (the fourth of 24), being 4 times smaller than the product of 47 by 24, it follows, that if we render the multiplier 4 times as small, or if we divide it by 4, the product is rendered 4 times as small by this change.

We have seen, moreover (26), that in a multiplication of two factors, we can invert the order of the factors; then, what we have just said with reference to the multiplier, applies equally to the multiplicand; then,

It results from this, that we do not change a product, in rendering the multiplicand a certain number of times greater, provided we render at the same time the multiplier a certain number of times less, (*i. e.*) by multiplying one factor by a certain number, and dividing the other factor by the same number.

It is upon this last consequence that we found a method which is sometimes employed to verify multiplication.

Given, to multiply 347 by 72. To multiply 347 by 72, is to multiply twice 347, or 694, by the half of 72, or by 36. Thus, after having multiplied 347 by 72, we can then multiply 694 by 36; and, if the operations are correct, we ought to find the same result.

Now, since in division the dividend is a product, of which the divisor and the quotient are two factors, it follows that if we multiply or divide the dividend by a certain entire number, the quotient is by this change multiplied or divided by the same entire number.

For, as, after this change, the quotient, multiplied by the divisor, must produce a dividend a certain number of times greater or less than the first dividend, it follows necessarily, the divisor remaining the same, that the quotient must be the same number of times greater or less.

On the contrary, if, without altering the dividend, we render the divisor a certain number of times greater or smaller, the quotient is thereby rendered the same number of times smaller or greater. This is the sole admissible hypothesis, in order that the multiplication may give the same product or the same dividend.

Then, by multiplying or dividing the dividend and the divisor by the same number, we do not change the quotient; since, if, by the change of the dividend, we multiply or divide the quotient by a certain number, the second change renders it the same number of times smaller or greater. Thus, the compensation leaves it the same.

Exercises on this Chapter.

1. Enunciate the number 10030047089500476.

2. Enunciate the same number, the middle figure being left out.

3. How many figures in a number, the first figure on the left of which expresses hundreds of septillions?

4. What is wanting in the number 2047035007, in order to form unity, followed by as many zeros as there are figures in the number?

5. Being given to subtract 58900564 from 62080347, if we substitute for the smaller number that which would make it unity followed by 8 zeros, and if we add this complement to the greater number, what must we then do to obtain a result equal to that which we obtain by the direct operation?

6. The day being composed of 24 hours, the hour of 60 minutes, the minute of 60 seconds; required how many seconds there are in the year, which we suppose to have exactly 365 days.

7. Required what changes would be made in the product of 67084 by 3769, by supposing, 1st, That the multiplier is augmented by 10, the multiplicand remaining the same? 2d, That the multiplicand is augmented by 10, the multiplier remaining the same? 3d, That the two factors are simultaneously augmented by 10 units?

8. What is the population of a county containing 16537 square miles, each square mile averaging 45 inhabitants?

9. The light of the sun reaches the earth in 8 minutes and 13 seconds, and the distance is 95000000 miles. How far does the light travel in one second?

CHAPTER II.

FRACTIONS.

41. We have already seen (Nos. 1 and 8), what a fraction is, and what idea we are to form of it. We distinguish always two terms in a fraction, the denominator and the numerator. The denominator indicates into how many equal parts unity is divided, and the numerator how many of these parts are taken. These two, taken together, constitute the fraction.

Thus, in the fraction $\frac{3}{4}$, which we call three-fourths, 4 is the denominator, and shows that the unit is divided into 4 equal parts; 3 is the numerator, and indicates that we take 3 of these parts. In the same manner, the fraction $\frac{11}{12}$, eleven-twelfths, expresses that the unit is divided into 12 equal parts, and that we take eleven of them. We have seen, also, that such a fraction as $\frac{13}{15}$ is equivalent to the 15th part of the whole number expressed by 13; that is to say, a fraction can also be considered as the quotient of its numerator divided by its denominator, so that thirteen times the 15th part of unity, or thirteen-fifteenths, and the fifteenth part of thirteen, or thirteen divided by fifteen, are identical expressions.

This last point of view leads us naturally to consider fractional expressions, such as $\frac{19}{6}$, $\frac{23}{12}$, $\frac{47}{15}$, ... of which the numerator is greater than the denominator.

Now, these expressions are easily comprehended, as they result from the division of the numbers 19, 23, 47, respectively into 6, 12, 15, equal parts.

But how can $\frac{19}{6}$ express 19 times the 6th part of unity?

For this we conceive that we have four principal units, of which each one is divided into 6 sixths; then, in order to form 19, or $6\times3+1$ of them, we take the 18 sixths, of which the three first principal units are composed, and add to them one of the parts of the fourth principal unit.

We obtain thus 19 sixths, or $\frac{19}{6}$.

By extending this principle, we can place unity or any entire number under a fractional form.

Thus, 1 can be written $\frac{12}{12}$, $\frac{15}{15}$, &c.

In the same manner, 10, 14, 25, &c., can be written $\frac{10}{1}$, $\frac{14}{1}$, $\frac{25}{1}$, &c.

42. From the definition which we have just given of the numerator and denominator of a fraction, the following consequences obviously result.

1st. *If we multiply or divide the numerator of a fraction by a number, the denominator remaining the same, the new fraction will be this number of times greater or less than the first.*

For, when we multiply the numerator by 2, 3, 4, ... we indicate thereby, that we take 2, 3, 4, ... more parts; and as the parts are the same, the new fraction is 2, 3, 4, ... times greater. Thus, let the fraction be $\frac{6}{25}$; it is clear that $\frac{12}{25}$, $\frac{18}{25}$, $\frac{24}{25}$, ... are fractions 2, 3, 4, ... times greater than the first. Again, in dividing the numerator by 2, 3, 4 . . . . we take 2, 3, 4 . . . . times fewer parts than, &c. . . . . Thus, $\frac{3}{25}$, $\frac{2}{25}$, are 2, 3, times smaller than $\frac{6}{25}$.

2d. *If we multiply or divide the denominator of a fraction by a number, the numerator remaining unchanged, we divide or multiply the fraction by this number*

For, when we multiply the denominator by 2, 3, 4 . . . . we indicate that the unit is divided into parts 2, 3, 4 . . . . times more parts. Each of these parts is then 2, 3, 4 . . . . times smaller; and as we take always the same number of these parts, it follows that the fraction is 2, 3, 4 . . . . times smaller.

If we divide the denominator by 2, 3, 4 . . . . the unit is divided into 2, 3, 4 . . . . times fewer parts; each one of these parts is then 2, 3, 4 . . . . times greater; therefore, &c.

3d. *We do not change the value of a fraction by multiplying or dividing its two terms by the same number.*

For the change made in the numerator renders the fraction a certain number of times greater or smaller; but the same change made in the denominator renders it on the contrary the same number of times smaller or greater; then one change compensates the other, and the value of the fraction is not changed.

Thus, the fractions $\frac{6}{8}$, $\frac{9}{12}$, $\frac{12}{16}$, $\frac{15}{20}$, ... are all equivalent to the fraction $\frac{3}{4}$, since they result from the multiplication of each of the terms of the latter by 2, 3, 4, 5. In the same way, the fraction $\frac{24}{36}$ is equal to the fraction $\frac{12}{18}$, or $\frac{8}{12}$, or $\frac{6}{9}$, since we obtain these last by dividing the two terms of $\frac{24}{36}$ by 2, 3, 4.

These different propositions can also be considered as consequences of the second manner of viewing a fraction (see No. 41), and the principles established (No. 40), in division.

43. As the third proposition is of continual application, we will give a demonstration of it, direct and independent of the two first.

Take, for example, the fraction $\frac{5}{8}$, and multiply the two terms 5 and 8 by 3, which gives $\frac{15}{24}$. We have to prove that this last fraction is equivalent to the first.

For, the principal unit being divided at first into eight equal parts, let us divide each eighth into three equal parts; the unit is thus divided into twenty-four equal parts.

Each eighth equals, then, three twenty-fourths, and five-eighths equal five times three, or fifteen twenty-fourths; that is to say, the fractions $\frac{5}{8}$ and $\frac{15}{24}$ have absolutely the same value.

We would demonstrate, in the same manner, that the fractions $\frac{11}{12}$ and $\frac{55}{60}$, the latter of which is formed by multiplying the two terms, 11 and 12, of the first, by 5, are equal.

As reciprocally we pass from the fraction $\frac{15}{24}$ to the fraction $\frac{5}{8}$, by taking the third of each term, and from the fraction $\frac{55}{60}$, to the fraction $\frac{11}{12}$, by taking the fifth of the two terms of the former, we can conclude that *a fraction does not change its value when we multiply or divide its two terms by the same number.*

Let us pass to the different operations which we may have to perform on fractions in the resolution of a question, the data of which are fractions or fractional numbers.

But, before explaining the four fundamental operations, we will make known two transformations of frequent use in the calculus of fractions.

REDUCTION OF FRACTIONS TO THE SAME DENOMINATOR.

44. This transformation has for its object to *reduce two or more fractions, having different denominators, to the same denominator.* Now, the principle that we do not change the value of a fraction by multiplying its two terms by the same number, furnishes a simple means of effecting this transformation.

Let, for example, $\frac{3}{4}$ and $\frac{5}{7}$ be the fractions which are to be reduced to the same denominator.

If we multiply the two terms, 3 and 4, by 7, the denominator of the second, and the two terms, 5 and 7, of the second, by 4, the denominator of the first, there will result $\frac{21}{28}$ and $\frac{20}{28}$ for the two fractions required.

These fractions have the same value as the fractions proposed according to the principle of (43). Again, they have necessarily equal denominators, since each one of them comes from the multiplication of the two primitive denominators, 4 and 7, by each other.

Again, given the fractions, $\frac{4}{7}$, $\frac{5}{8}$, $\frac{6}{11}$, to be reduced to the same denominator.

Multiply the two terms, 4 and 7, of the first fraction, by 88, product of the denominators, 8 and 11, of the second and third; then the two terms, 5 and 8, of the second, by 77, product of the denominators, 7 and 11, of the first and third; finally, the two terms, 6 and 11, of the third, by 56, product of the denominators of the first and second. We will thus obtain the new fractions, $\frac{352}{616}$, $\frac{385}{616}$, $\frac{336}{616}$.

These fractions have the same value as the primitive fractions, and their denominators are necessarily the same, since each one of them is the product of the denominator of each fraction by the product of the two other denominators.

General Rule. — *In order to reduce any number whatever of fractions to the same denominator, multiply successively the two terms of each one of them by the product of the denominators of the other fractions.*

We will show the method of applying this rule in practice.

Let the fractions be $\frac{3}{8}$, $\frac{7}{11}$, $\frac{10}{13}$, $\frac{23}{25}$, and $\frac{29}{43}$.

For greater simplicity we arrange the operation thus:

$\frac{3}{8}$	$\frac{7}{11}$	$\frac{10}{13}$	$\frac{23}{25}$	$\frac{29}{43}$
153725	111800	94600	49192	28600
$\frac{461175}{1229800}$,	$\frac{782600}{1229800}$,	$\frac{946000}{1229800}$,	$\frac{1131416}{1229800}$,	$\frac{829400}{1229800}$.

After having formed the product of the five denominators, 8, 11, 13, 25, and 43, which gives for the common denominator of the transformed fractions, 1229800, we divide successively this product by each one of the denominators, and we obtain the five quotients, 153725, 111800, 94600, 49192, 28600, which we place respectively below the five proposed fractions; after which, we multiply the numerator of each fraction by the quotient which corresponds to it; and we obtain thus the different numerators.

As to the common denominator, it is, as we have said above, equal to 1229800.

The reason of this manner of proceeding is easily perceived, for the number, 1229800, being the product of the five denominators, the quotient, 153725, of the division of 1229800 by 8, expresses necessarily the product of the four other denominators, 11, 13, 25, 43.

In the same manner, 111800, being the quotient of the division of 1229800, by the second denominator, 11, is equal to the product of the four other denominators, 8, 13, 25, and 43; and the same reasoning applies to the other quotients. This method is, moreover, much more expeditious, than if, for each fraction, we performed the multiplication of the denominators of the four others. But it is only really advantageous when there are more than three fractions to be reduced to the same denominator.

45. There is a case in which the reduction to the same denominator can be performed in a very simple manner; that is, when the greatest of the denominators is exactly divisible by each one of the others.

Let the fractions be, for example,

$$\begin{array}{ccccc} \frac{2}{3} & \frac{3}{4} & \frac{5}{6} & \frac{7}{12} & \frac{23}{36} \\ 12 & 9 & 6 & 3 & 1 \\ \frac{24}{36}, & \frac{27}{36}, & \frac{30}{36}, & \frac{21}{36}, & \frac{23}{36}. \end{array}$$

It is easy to see that 36, divisible by itself, is also divisible by each one of the four other denominators, 3, 4, 6, and 12.

This being fixed, we effect successively these divisions, and place the quotients, 12, 9, 6, 31, below the four first fractions; after which, we multiply the numerator of each one of them by the quotient which corresponds to it; the fraction, $\frac{23}{36}$, remains as it was, and all the fractions are reduced to the denominator, 36.

Sometimes, although the greatest denominator is not divisible by all the others, we perceive that, by multiplying it by 2, 3, 4 . . . . we obtain a product exactly divisible by all the denominators. This affords us, likewise, a means of simplification.

Let the fractions be,

$$\begin{array}{cccccc} \frac{3}{4} & \frac{7}{8} & \frac{11}{12} & \frac{13}{18} & \frac{17}{24} & \frac{25}{36} \\ 18 & 9 & 6 & 4 & 3 & 2 \\ \frac{54}{72} & \frac{63}{72} & \frac{66}{72} & \frac{52}{72} & \frac{51}{72} & \frac{50}{72} \end{array}$$

The denominator, 36, is divisible separately by 4, 12, and 18, but is not divisible by 8 nor by 24; but, if we double it, we obtain 72, a number exactly divisible by each one of the denominators.

This being fixed, we form the quotients of 72 by each one of the denominators, and place them respectively below the fractions; we then multiply the numerator of each one of them by the quotient which corresponds to it; all these fractions will have 72 for common denominator.

FORMATION OF THE LEAST COMMON DENOMINATOR OF SEVERAL FRACTIONS.

46. The simplifications which we have just explained, require some practice to see when they can be applied; but there is a direct means of obtaining, in all cases, the *Least Common Denominator* of several fractions.

To do this, we must find the *least common multiple* of the denominators; that is, the least number divisible by all of them. To do this, decompose the numbers into their smallest possible factors; that is, *prime* factors, or factors divisible only by themselves and unity. Then form the product of all these prime factors, common or not common, to the numbers. We obtain thus a result, evidently divisible by all the numbers; and it is, besides, the smallest number so divisible; for, any number containing one of the *prime* factors a smaller number of times than one of the given numbers, would not be divisible by that one of these numbers which contained this factor a greater number of times. (A more thorough discussion of this we will give under the chapter on the properties of numbers).

Applying the above to the last example, we have,

4	8	12	18	24	36
2.2	2.2.2	2.2.3	2.3.3	2.2.2.3	2.2.3.3

Having thus arranged the numbers and their prime factors, we see that 2.2.2.3.3 is evidently the least common multiple. Performing the multiplication, we obtain 72, as before.

Let the fractions be for a new example,

$$\frac{11}{15} \quad \frac{13}{18} \quad \frac{17}{24} \quad \frac{25}{28} \quad \frac{37}{44} \quad \frac{83}{140} \quad \frac{29}{175} \quad \frac{233}{480}$$

the numerators of which do not contain, at least apparently, prime factors (as 2.3.5 . . . .), which may be, at the same time, contained in the corresponding denominators; otherwise, it would be necessary to suppress these factors in the two terms.

Recomposing the denominators, we find for results,

3.5 | 2.3.3 | 2.2.2.3 | 2.2.7 | 2.2.11 | 2.2.5.7 | 7.5.5 | 2.2.2.2.2.5.3;

which gives for the least common multiple,

2.2.2.2.2.3.3.5.5.7.11, or 554400;

which is the least common denominator to be given to all the fractions; a number far less than that which we would obtain by applying the general rule in No. (44).

Nothing more remains now but to determine the numbers by which we are to multiply the numerators, in order to obtain the numerators of the new fractions; and for this it is necessary, as we have already seen, to divide 554400 by each one of the given denominators.

Relations of Magnitude among Several Fractions.

We have here some applications of the preceding transformations.

47. Question 1st. — *Of the two fractions,* $\frac{3}{5}$ *and* $\frac{7}{12}$, *which is the greater?*

We cannot, at first sight, answer this question; because, though on the one hand the unit in the second fraction is divided into a greater number of parts than in the first, on the other hand, we take more of these parts, since the numerator, 7, is greater than 3.

But we remove the difficulty by reducing them to the same denominator; for it is evident that of two fractions which have the same denominator, the greater is that which has the greater numerator. This reduction effected, we obtain $\frac{36}{60}$ for the first fraction, and $\frac{35}{60}$ for the second; the fraction, $\frac{3}{5}$, is the greater of the two.

We find, in the same manner, that of the three fractions, $\frac{4}{7}$, $\frac{6}{11}$, $\frac{8}{13}$, the greatest is $\frac{8}{13}$, the smallest $\frac{6}{11}$; for, being reduced to the same denominator, they become, respectively, $\frac{572}{1001}$, $\frac{546}{1001}$, $\frac{616}{1001}$.

We could equally well reduce the fractions to the same numerator (by applying to the numerators what has been said concerning the denominators); and of these fractions the greatest would be that which would have the smallest denominator; since, the parts being greater, we take the same number of them. But the first method has the advantage of making known, at the same time, the differences which exist between the fractions, compared two and two.

48. Question 2d. — *What change do we produce in a fraction, by adding the same number to its two terms?*

Let the fraction be $\frac{7}{12}$, for example, to both terms of which we add 6; $\frac{13}{18}$ is the resulting fraction.

If now we reduce these two fractions to the same denominator, the first becomes $\frac{126}{216}$, and the second $\frac{156}{216}$. The proposed fraction is then increased in value. In order to give a reason for this fact, we observe that, unity being equal to $\frac{12}{12}$, the excess of unity above $\frac{7}{12}$ is expressed by $\frac{5}{12}$; in the same manner, the excess of unity above $\frac{13}{18}$ is expressed by $\frac{5}{18}$. The numerators of these two differences are the same, which should be the case; for, 18 and 13, having been formed by the addition of 6 to the two terms, 7 and 12, it follows, that there is the same difference between 18 and 13, as between 7 and 12. But the difference, $\frac{5}{18}$, is necessarily less than the difference, $\frac{5}{12}$, since the first denominator is the greater, and the numerators are equal; then the fraction, $\frac{13}{18}$, differs less from unity than the fraction, $\frac{7}{12}$; consequently, the first is greater than the second.

We see, moreover, that the greater the number added to the two terms of the fraction, $\frac{7}{12}$, the smaller the difference between unity and the new fraction; since the numerator of this difference, being always 5, the denominator becomes greater and greater. As this same reasoning can be applied to every other fraction, we can draw the conclusion that if to the two terms of a fraction we add the same number, the resulting fraction is greater than the given fraction; and it is greater, the greater the number added.

Conversely, by the same reasoning, a fraction is diminished in value when we subtract the same number from its two terms.

N. B. The contrary would take place, if the fractional number was greater than unity, as $\frac{17}{14}$.

Adding 8 to the two terms, we would have $\frac{25}{22}$ less than $\frac{17}{14}$. For, $\frac{25}{22}$ exceeds unity by $\frac{3}{22}$ only, while $\frac{17}{14}$ surpasses unity by $\frac{3}{14}$, greater than $\frac{3}{22}$.

We have thought it necessary to enter into some details upon this proposition, in order to prevent beginners from confounding this with (43), when we multiply or divide the two terms of a fraction by the same number.

Reduction of a Fraction to its Simplest Terms.

49. It happens often, in the calculus of fractions, that we are led to a fraction expressed by large numbers; now, the greater the numerator and denominator, the greater trouble we have to form a just idea of the fraction.

For example, the fraction, $\frac{12}{15}$, indicates, that we must divide unity into 15 equal parts, and take 12 of these parts. But 12 and 15 being, at the same time, divisible by 3, if we perform the divisions, there results $\frac{4}{5}$, a fraction equivalent to the one given; then, in order to form an idea of it, it suffices to conceive the unit divided into 5 equal parts, and to take 4 of them, which is much simpler. When then we have a fraction, the terms of which are quite large, it is best to reduce it, if possible, to a fraction whose terms are smaller.

The first method which presents itself is to divide the two terms by the numbers, 2, 3, 4 . . . . as long as that is possible.

1st. *Let the fraction*, $\frac{108}{144}$, *be given.* The two terms of this fraction are evidently divisible by 4; and, in effecting the division, we obtain $\frac{27}{36}$; but the two terms of this are divisible by 9; and this new division gives for a result, $\frac{3}{4}$, which cannot be farther reduced.

This example presents no difficulty; but this is not always the case, especially when the two terms of the given fraction are composed of three or more figures; for it can happen that a prime factor of two or three figures is common to the two terms of the fraction, without our being able to find it by mere inspection. Hence, we see the necessity of having a general method of reducing a given fraction to the most simple expression possible. This method we will now discuss. It is called the method of the *greatest common divisor.*

50. We commence by establishing several preliminary notions.

A number is called the *multiple* of another number, when it contains it a certain number of times, as we have already seen.

Reciprocally, the second number is called a *submultiple,* or an *aliquot part,* or simply a *divisor* of the first.

We call a *prime number* a number which is only divisible by itself, and by unity, which is a divisor of every number. Thus, 2, 3, 5, 7, 11, 13 . . . . are prime numbers; but 4, 6, 8, 9, 12, are not prime numbers; since they have the divisors, 2 and 3. Two numbers are said to be *prime* with respect to each other, when they have no other common divisor except unity; thus, 4 and 9, 7, 1, and 12, are numbers prime with respect to each other; 8 and 12 are not, since they are divisible at the same time either by 2 or 4.

First Principle. — *Every number, which exactly divides another number, divides also any multiple whatever of this second number.*

For example, 24 being divisible by 8, and giving for a quotient 3, 5 times 24, or 120, divided by 8, will give (No. 40) for quotient, 5 times 3, or 15. In the same manner, 60 being divisible by 12, and giving for quotient 5, 7 times 60, or 420, divided by 12, will give for quotient 7 times 5, or 35.

Second Principle. — *Every number, decomposed into two parts, both divisible by a second number, is itself divisible by* *this second number.* For the quotient of the division of the total being equal to the sum of the two partial quotients, if these two partial quotients are entire, their sum, or the total quotient, must be entire.

Third Principle.—Every number which divides exactly a whole, decomposed into two parts, and which divides one of these parts, ought to divide also the other part. For the total quotient being equal to the sum of the two partial quotients, if one of these partial quotients is fractional, it would follow that an entire number would be equal to a fractional number; which would be absurd.

51. So much being established, *let the two numbers*, 360 *and* 276, *be given, of which we propose to determine the greatest common divisor*, or the greatest number which will divide them both exactly. It is at once evident that this greatest common divisor cannot exceed the smaller number, 276; and as 276 divides itself, it follows, that if it will divide 360 also, it will be the greatest common divisor sought. Attempting this division of 360 by 276, we find for quotient, 1, and remainder, 84; then, 276 is not the greatest common divisor. We say, now, that the greatest common divisor of 360 and 276, is the same as that which exists between the smaller number, 276, and the remainder of the division.

For the greatest common divisor sought, since it ought to divide 360, and one of its parts, 276, divides necessarily the other part, 84, (50); whence, we can conclude at once, that the greatest common divisor of 360 and 276, cannot exceed that which exists between 276 and 84; since it must divide these two numbers. In the second place, the G. C. D. of 276 and 84, dividing the two parts, 276 and 84, of 360, divides necessarily this number; being the exact divisor of 360 and 276, it cannot exceed the greatest C. D. of 360 and 276. Whence, we see, that the G. C. D. of 360 and 276, and the G. C. D. of 276 and 84, cannot be greater than each other; then they are equal.

Thus, the question is reduced to seeking the greatest C. D. of 276 and 84; numbers simpler than 360 and 276.

We now reason on 276 and 84, as we have about the primitive numbers; that is, we try the division of 276 by 84; because, if the division is exact, 84 will be the G. C. D. of 276 and 84, and consequently of 360 and 276.

Effecting this new division, we have 3 for quotient, and 24 for remainder; then 84 is not the G. C. D. sought. But, by analogous reasoning to that above, we can prove that the G. C. D. of 276 and 84 is the same as that of the first remainder, 84, and the second remainder, 24. The question is then reduced to finding the G. C. D. of 84 and 24; we divide 84 by 24, and obtain 3 for quotient, and 12 for remainder; then 24 is not the G. C. D.; but, as this G. C. D. is the same as that of 24 and 12, we divide 24 by 12; we find an exact quotient, 2; thus, 12 is the greatest C. D. of 12 and 24, hence of 84 and 24, of 276 and 84, and, finally, of 360 and 276, or the G. C. D. sought.

In practice, we arrange the operation thus:

$$\begin{array}{c|c|c|c|c}
 & 1 & 3 & 3 & 2 \\
\underline{360} & \underline{276} & \underline{84} & \underline{24} & 12 \\
84 & 24 & 12 & 0 &
\end{array}$$

After having divided 360 by 276, we place the quotient, 1, above the divisor, and a remainder, 84; we write this remainder to the right of the less number, 276, and we divide 276 by 84, placing the quotient, 3, above the divisor, and the remainder, 24, to the right of the 84, and so on for the rest.

General Rule. — *In order to find the G. C. D. of two numbers, divide the greater number by the less; if there is no remainder, the smaller number is the G. C. D.*

If there is a remainder, divide the less number by this remainder; and if this division is without remainder, the first remainder is the G. C. D.

If this second division gives a remainder, divide the first remainder by the second, and continue always to divide the pre-

ceding remainder by the last remainder, until the division becomes exact; then the last divisor employed will be the G. C. D. sought.

If the last divisor is unity, it is a proof that the two numbers are prime with respect to each other. Reciprocally, if two numbers are prime with each other, and if we apply the method above, we will find necessarily a final remainder equal to unity. For, according to the nature of the method, the remainders go on diminishing; besides, we cannot obtain a remainder, nothing, before having obtained a remainder, 1; since the divisor, different from unity, which gave this remainder nothing, would be the common divisor of the two numbers. Thus, we must, necessarily, after a smaller or greater number of operations, obtain unity for a remainder.

52. We give now the application of this method.

Reduce the fraction, $\frac{592}{999}$, to its simplest form.

$$\begin{array}{c|c|c|c|l}
 & 1 & 1 & 2 & \\
999 & 592 & 407 & 185 & \frac{5}{37} \\
\overline{407} & \overline{185} & \overline{37} & \overline{00} &
\end{array}$$

$$\begin{array}{c|c|c|c}
999 & 37 & 592 & 37 \\
\overline{259} & \overline{27} & \overline{222} & \overline{16} \\
\overline{0} & & \overline{00} &
\end{array}$$

We find, for the greatest common divisor, 37, and, dividing 999, and 592, by 37, we have $\frac{16}{27}$, for the value of $\frac{592}{999}$, reduced to its least terms.

If we can find no common divisor greater than unity for the terms of the fraction, the fraction is irreducible, its terms being prime with each other.

Remark. — If, in the operation for the common divisor, we arrive at a prime number for a remainder, as, for example, 5 or 7, we can conclude at once that unless this prime remainder exactly divides the last divisor, the two primitive numbers have no common divisor greater than unity. The reason of this is obvious. We will return once more to the operation of the greatest common divisor, as it is one of the most important in the arithmetic.

Second example, $\frac{29003}{36569}$. We find for G. C. D. 1261, respective quotients, 29 and 23; thus, $\frac{23}{29}$ is the fraction reduced to its simplest terms.

53. From what precedes, it results, that if from the two terms of a fraction, we subtract the same multiples of the two terms of the equivalent irreducible fraction, the resulting fraction is also equivalent to the given one.

Let us take, for example, the fraction, $\frac{18}{24}$, which, reduced to its least terms, according to the method indicated in the preceding article, is equal to $\frac{3}{4}$. If, from the two terms, 18 and 24, of the given fraction, we subtract four times 3, or 12, and four times 4, or 16, we obtain a new fraction, $\frac{6}{8}$, which, expressed in simpler terms than those of the given fraction, is equal to it. For, in suppressing the factor, 2, common to 6 and 8, we find $\frac{3}{4}$, as for the first fraction, $\frac{18}{24}$.

It is easy to explain this result. For, if $\frac{18}{24}$ is equal to $\frac{3}{4}$, an irreducible fraction, of which the two terms are prime with each other, 18 and 24 must be the same multiples (6 times 3, and 6 times 4), of the two terms of the fraction, $\frac{3}{4}$; and when from 18 and 24 we subtract four times 3, and four times 4, we obtain differences, twice 3, and twice 4, which are also the same multiples of 3 and 4; whence results a new fraction, $\frac{6}{8}$, equal to $\frac{3}{4}$. It would seem that this proposition ought to furnish a means of simplifying a fraction; but we see that this means would be altogether illusory, since it supposes the irreducible fraction known, to which the given one is equivalent.

N. B. We would remark here, that we subtract from the two terms of the fraction two different numbers, and not the same number as in (48).

We pass, now, to the four fundamental operations upon fractions.

ADDITION OF FRACTIONS.

54. The addition of fractions has for its object *to find a single fraction which shall express the value of the sum of several fractions.*

There are two cases; the fractions to be added are either of the *same species,* that is to say, have the *same denominator,* or of *different species.*

In the first case, *we sum up the numerators, and then give to this sum the common denominator.*

In the second case, *we reduce the fractions to the same denominator; after which we operate as in the first case.* The reason is obvious, since the denominator is a sign indicating the value or species of the units to be added, and the numerator the number of these units.

$$\text{Thus, } \frac{2}{11}+\frac{3}{11}+\frac{4}{11}=\frac{2+3+4}{11}=\frac{9}{11}$$

In the same manner,

$$\frac{5}{23}+\frac{2}{23}+\frac{7}{23}+\frac{4}{23}=\frac{5+2+7+4}{23}=\frac{18}{23}$$

Let it be given, now, to add the three fractions,

$$\begin{array}{ccc} \frac{2}{3} & \frac{3}{4} & \frac{7}{8} \\ 8 & 6 & 3 \\ \frac{16}{24} & \frac{18}{24} & \frac{21}{24} \end{array}$$

After having reduced these fractions to the least common denominator, 24, (No. 46), we add the numerators, 16, 18, and 21; we then give to the sum, 55, the denominator, 24.

We have thus $\frac{55}{24}$ for the result required.

55. This last example leads to a fractional expression, $\frac{55}{24}$, greater than unity, which gives rise to a new operation.

We have seen that unity is equivalent to $\frac{24}{24}$, or twenty-four twenty-fourths; whence, it follows, that as many times as 55 contains 24, so many units there are in $\frac{55}{24}$. Now, dividing 55 by 24, we have for a quotient, 2, with remainder, 7; thus, $\frac{52}{24}$ is a number composed of 2 units and $\frac{7}{24}$. In general, when we obtain a fractional result, of which the numerator exceeds the denominator, in order to extract the whole number contained in this expression, *we must divide the numerator by the denominator. The quotient which we obtain represents the entire number, and the remainder is the numerator of the fraction which is to be added to the entire number,* (43).

By this mode, we find,

$$\tfrac{17}{12}=1\tfrac{5}{12};\ \ \tfrac{153}{15}=10\tfrac{3}{15}=10\tfrac{1}{5};\ \ \tfrac{654}{89}=7\tfrac{31}{89}.$$

Reciprocally, when we have an entire number joined to a fraction, in order to form a single fractional number, *we must multiply the entire number by the denominator, add the product to the numerator, and give to the sum the denominator of the fraction.*

For example,

$$3\tfrac{2}{5}=\frac{3\times 5+2}{5}=\frac{17}{5};\ \ 11\tfrac{7}{12}=\frac{11\times 12+7}{12}=\frac{139}{12};\ \text{\&c.}$$

SUBTRACTION OF FRACTIONS.

56. The subtraction of fractions has for its object *to find the excess of a greater fraction over a smaller.*

If the two fractions have the *same* denominator, *we subtract the smaller numerator from the greater, and give to the difference the common denominator.*

If they have not the same denominator, we reduce them to such as have; after which, we proceed as in the first case.

Given, to subtract $\frac{5}{12}$ from $\frac{11}{12}$; there remain $\frac{6}{12}$, or $\frac{1}{2}$. In the same manner, $\frac{17}{24}-\frac{7}{24}=\frac{10}{24}=\frac{5}{12}$.

Given, to subtract $\frac{2}{3}$ from $\frac{7}{8}$. These two fractions give $\frac{16}{24}$ and $\frac{21}{24}$, by reduction to the same denominator; and we have

$$\frac{21}{24}-\frac{16}{24}=\frac{21-16}{24}=\frac{5}{24}.$$

$$\text{Also, }\frac{19}{20}-\frac{13}{17}=\frac{19\times 17-13\times 20}{20\times 17}=\frac{63}{340}.$$

We can have an entire number joined to a fraction, to be subtracted from an entire number joined to a fraction; or, as they are called, a *mixed number*, to be subtracted from a *mixed number*.

Given, for example, to subtract $5\frac{11}{13}$ from $12\frac{3}{4}$.

$$\begin{array}{r}
12\frac{3}{4}=12\frac{39}{52}=11\frac{91}{52} \\
5\frac{11}{13}=\ 5\frac{44}{52}=\ 5\frac{44}{52} \\
\hline
6\frac{47}{52}.
\end{array}$$

We commence by reducing the two fractions to the same denominators, which gives $\frac{39}{52}$ for the first, and $\frac{44}{52}$ for the second.

Then, as we cannot subtract $\frac{44}{52}$ from $\frac{39}{52}$, we take from the entire part, 12 of the greater number, one unit, which we add to the $\frac{39}{52}$, making $\frac{91}{52}$; we then subtract $\frac{44}{52}$ from $\frac{91}{52}$, and have for a remainder, $\frac{47}{52}$. Passing to the subtraction of the entire numbers, we regard the greater number as diminished by unity, and subtract 5 from 11, which gives 6. We have thus $6\frac{47}{52}$ for the required result.

The same result could be obtained by reducing the mixed numbers to single fractions by last article, and then following the rule given for subtraction of fractions.

MULTIPLICATION OF FRACTIONS.

57. Multiplication has for its object in general, *two numbers being given* to form a third number, which is compounded with the first number in the same manner as the second number is compounded with unity.

This being established, we distinguish three principal cases in the multiplication of fractions. We can have,

1st. *A fraction to be multiplied by an entire number.*

Given, for example, $\frac{7}{12}$, to be multiplied by 5.

According to the definition above, since the multiplier, 5, contains 5 times unity, it follows, that the product ought to be equal to 5 times $\frac{7}{12}$, or 5 times as great as $\frac{7}{12}$. Now, we have seen in

(43), that we render a fraction 5 times greater by multiplying its numerator by 5. We thus have $\frac{5 \text{ times } 7,}{12}$ or $\frac{35}{12}$, for the required product.

Then, *in order to multiply a fraction by an entire number, we must multiply the numerator by the entire number, and give to the product the denominator of the fraction.*

Given, to multiply $\frac{11}{18}$ by 9.

We obtain $\frac{99}{18}$ for the product, or $5\frac{9}{18}$, or $5\frac{1}{2}$. This result can be obtained more simply thus. For, by (43), we can divide the denominator by 9, instead of multiplying the numerator. And we find thus, $\frac{11}{2}$, or $5\frac{1}{2}$, for the required product.

We can only apply this last method, when the denominator is divisible by the number. The established rule is always applicable. Usage alone renders us familiar with these simplifications.

2d. *To multiply an entire number by a fraction.*

Example. — 12 to be multiplied by $\frac{4}{7}$.

Since, in this case, the multiplier, $\frac{4}{7}$, is equal to 4 times the 7th part of unity, the product ought to be equal to 4 times the 7th of 12. Now, the 7th of 12 is equal to $\frac{12}{7}$; and, in order to render this 4 times as great as $\frac{12}{7}$, we must multiply the numerator by 4; we thus obtain $\frac{48}{7}$, or $6\frac{6}{7}$, for the required product.

Then, *to multiply an entire number by a fraction, we multiply the entire number by the numerator, and give to the product the denominator of the fraction.*

$$\text{Thus, } 29 \times \frac{7}{8} = \frac{203}{8} = 25\tfrac{3}{8}.$$

$$24 \times \frac{5}{6} = \frac{120}{6} = 20.$$

We might find this last result by dividing 24 by 6, and multiplying the result by 5.

But, we repeat, these simplifications are not always possible.

3d. *A fraction to be multiplied by a fraction.*

Example. — Given, to multiply $\frac{3}{4}$ by $\frac{5}{8}$.

The reasoning is analogous to that of the preceding case; since $\frac{5}{8}$ is equal to 5 times the 8th part of unity, the product ought itself to be 5 times the 8th part of the multiplicand, $\frac{3}{4}$. Now, in order to take the 8th of $\frac{3}{4}$, we must (43) multiply the denominator by 8, which gives $\frac{3}{32}$; and in order to obtain a fraction 5 times as great as $\frac{3}{32}$, we must multiply the numerator by 5; which gives $\frac{15}{32}$ for the product required.

Then, *to multiply one fraction by another, multiply numerator by numerator, and denominator by denominator; then make the second product denominator of the first.*

We find, thus,

$$\frac{7}{12} \times \frac{5}{6} = \frac{35}{72}.$$

$$\text{And } \frac{8}{15} \times \frac{3}{4} = \frac{24}{60} = \frac{2}{5}.$$

N. B. In the two preceding cases, the product is always less than the multiplicand; and this ought to be the case, since the operation is really taking a part of the multiplicand indicated by the fractional multiplier.

58. Finally, one of the factors of the multiplication, or both of them, may be mixed numbers. These numbers are equivalent, respectively, to the *improper fractions*, (the fractions greater than unity being called *improper fractions*), $\frac{23}{3}$ and $\frac{47}{8}$; performing the multiolication of these by the rule above, we obtain $\frac{1081}{24}$, or $45\frac{1}{24}$.

We could effect this multiplication by parts; that is to say, multiply, first, 7 by 5, $\frac{2}{3}$ by 5, 7 by $\frac{7}{8}$, and $\frac{2}{3}$ by $\frac{7}{8}$; then add these four products; but this method is much the longest.

DIVISION OF FRACTIONS.

59. Division has for its object: *Given, the product of two factors, and one of the factors to determine the other.*

It results, obviously, from this definition, and from that of multiplication, that the first number, called dividend, is compounded with the third, called quotient, in the same manner that the divisor is compounded with unity.

This established, in the division as in the multiplication of fractions, we distinguish three principal cases.

1st. *To divide a fraction by an entire number.*

Given, for example, $\frac{5}{7}$, to be divided by 6. Since the divisor is 6 times unity, it follows, that the dividend, $\frac{5}{7}$, is equal to 6 times the required quotient; then, reciprocally, the quotient ought to be the 6th part of $\frac{5}{7}$. Now, in order to take the 6th part of a fraction, or to obtain one 6 times as small, we must (43) multiply the denominator by 6; thus, we obtain $\frac{5}{6 \text{ times } 7}$, or $\frac{5}{42}$, for the required quotient.

Then, to divide a fraction by an entire number, *multiply the denominator of the fraction by the entire number, leaving the numerator the same.*

$$\text{Thus, } \tfrac{11}{12} \text{ divided by } 8 = \frac{11}{12 \times 8} = \frac{11}{96}.$$

$$\text{In the same manner, } \frac{23}{30} \text{ divided by } 12 = \frac{23}{360}.$$

The quotient of $\frac{18}{25}$ by 6, is $\frac{18}{150}$; but we can effect the division of $\frac{18}{25}$ by 6, by taking the 6th of the numerator, which gives $\frac{3}{25}$; the same with $\frac{18}{150}$, when we suppress the factor, 6, common to the two terms. Then we add to the above rule, or *divide the numerator by the divisor, when that is possible.*

2d. *To divide an entire number by a fraction.*

Given, to divide 12 by $\frac{7}{9}$.

Since the divisor, $\frac{7}{9}$, is equal to 7 times the 9th part of unity, it follows, that the dividend is also equal to 7 times the 9th part of the required quotient. Then, taking the 7th of 12, which gives $\frac{12}{7}$, we will have the 9th of the quotient sought; and to obtain this quotient itself, we must take 9 times $\frac{12}{7}$, which is done by multiplying the numerator by 9; we thus obtain $\frac{9 \text{ times } 12}{7}$, or $\frac{108}{7}$, equal to $15\frac{3}{7}$.

Then, *in order to divide an entire number by a fraction, we must multiply the entire number by the denominator, and divide the product by the numerator.*

Or, we can say, as we have here multiplied 12 by $\frac{9}{7}$, *multiply the entire number by the fraction inverted.*

3d. *To divide a fraction by a fraction.*

Given, to divide $\frac{3}{5}$ by $\frac{8}{11}$.

The reasoning is like the preceding. The divisor, $\frac{8}{11}$, being 8 times the 11th part of unity, the dividend, $\frac{3}{5}$, is also equal to 8 times the 11th of the quotient; then, the 8th of $\frac{3}{5}$, or $\frac{3}{40}$, is the 11th of the quotient; and 11 times $\frac{3}{40}$, or $\frac{33}{40}$, is the quotient sought.

Then, *to divide a fraction by a fraction, we must multiply the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor; then make the second product the denominator of the first.*

Or, in simple terms, *multiply the dividend by the divisor with its terms inverted.*

Thus, $\frac{3}{4} \div \frac{5}{7} = \frac{3}{4}$ times $\frac{7}{5} = \frac{21}{20} = 1\frac{1}{20}$.

In the same manner,

$$\frac{23}{30} \div \frac{13}{15} = \frac{23}{30} \times \frac{15}{13} = \frac{23 \times 15}{30 \times 13} = \frac{345}{390}.$$

(We could have suppressed the factor, 15, obviously common to both terms of the product in this last example, before performing the multiplication).

N. B. Whenever, in the division of fractions, the divisor is less than unity, the quotient will be greater than the dividend. For this quotient results from the multiplication of the dividend by the divisor inverted, a number greater than unity.

60. Finally, if we have a mixed number, *we reduce first the entire parts to fractions, and then proceed as in the case above.*

Given, $12\frac{3}{4}$ to be divided by $6\frac{2}{3}$. We have

$$12\tfrac{3}{4} \div 6\tfrac{2}{3} = \tfrac{51}{4} \div \tfrac{20}{3} = \tfrac{51}{4} \times \tfrac{3}{20} = \tfrac{153}{80}.$$

In the same manner,

$$7\tfrac{8}{11} \div 15\tfrac{5}{8} = \tfrac{85}{11} \div \tfrac{125}{8} = \tfrac{85}{11} \times \tfrac{8}{125} = \tfrac{680}{1375}.$$

Remark.—The rules for the division and multiplication of fractions can be very readily deduced by regarding them as unperformed divisions.

Remark II.—It is evident that the division of fractions can give rise to fractions with fractional terms, or *complex fractions*, as they are sometimes called. We can have, for example,

$$\frac{\tfrac{2}{3}}{\tfrac{5}{7}}, \quad \frac{4\tfrac{1}{2}+\tfrac{3}{4}}{2\tfrac{1}{7}+\tfrac{5}{8}}, \quad \frac{\tfrac{3}{5} \text{ times } \tfrac{2}{3}}{4\tfrac{2}{3} \text{ times } \tfrac{6}{7}},$$

Which are reduced to fractions of two terms by performing all the operations indicated upon the separate fractions, according to known rules.

FRACTIONS OF FRACTIONS.

61. To the multiplication of fractions attaches itself another species of operation, known under the name of the *rule for fractions of fractions*, or *compound fractions.*

In order to give an accurate idea of this operation, suppose, first, that we have to take a part of the fraction, $\frac{5}{7}$, indicated by the fraction, $\frac{2}{3}$. As this is the same thing as taking twice the third of $\frac{5}{7}$, or (57), multiplying $\frac{5}{7}$ by $\frac{2}{3}$, we have for result,

$$\frac{5 \text{ times } 2}{7 \text{ times } 3}, \text{ or } \frac{10}{21}.$$

Suppose, now, we wish to take a part of $\frac{10}{21}$, indicated by the fraction, $\frac{8}{13}$; we would have, as above, $\frac{8\times 10}{13\times 21}=\frac{80}{273}$, and this last expression would represent $\frac{8}{13}$ of $\frac{2}{3}$ of $\frac{5}{7}$.

Hence, we see, that in order to take fractions of fractions, *we must multiply all the numerators together, and all the denominators together, and give the last product as denominator to the first.*

When we have to take fractions of fractions of a given entire number, we put this entire number under the form of a fraction, having 1 for denominator, and apply the rule which has just been established.

Thus, the $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{8}$ of $\frac{6}{7}$ of $\frac{12}{1}=\frac{2\cdot 3\cdot 5\cdot 6\cdot 12}{3\cdot 4\cdot 8\cdot 7\cdot\ 1}=\frac{2160}{672}$, or, reducing, $3\frac{144}{672}=3\frac{3}{14}$.

We can simplify these, and similar operations, by suppressing the factors common to both terms.

Thus, in the example, $\frac{3}{4}$ of $\frac{5}{6}$ of $\frac{7}{12}$ of $\frac{6}{7}$ of $\frac{24}{1}$, if we suppress the common factors, we have

$$\frac{3\times 5}{2}, \text{ or } \frac{15}{2}, \text{ or } 7\tfrac{1}{2}.$$

APPROXIMATIVE VALUATION OF VULGAR FRACTIONS.

62. In order to complete the general theory of fractions, we will resolve the following question, which has many useful applications.

Given, an irreducible fraction, of which the terms are so large, that it is difficult to form an accurate idea of its value, to replace it by another which approaches it in value to within certain limits, but whose terms are much more simple; that is, which have for denominators, 2, 3, 4, 5, 6, &c.

Take, for example, the fraction, $\frac{523}{949}$. We propose to find its approximate value in *twelfths*, (*i. e.*) to replace it by a fraction having 12 for denominator.

We remark, first, that unity being equal to $\frac{12}{12}$, $\frac{523}{949}$ of unity are equal to $\frac{523}{949}$ of $\frac{12}{12}$, or equal to $\frac{523 \times 12}{949 \times 12}$. Multiplying 523 by 12, we obtain 6276; which, divided by 949, gives 6 for quotient, and 582 for remainder. Then, the fraction is $\frac{6}{12}$, with the remainder, $\frac{582}{949 \times 12}$, less than $\frac{1}{12}$. Hence, $\frac{6}{12}$ is the value of the fraction to within less than $\frac{1}{12}$.

63. In general, in order to transform a fraction, $\frac{a}{b}$, into another having a denominator, n, at the same time differing from the first by less than $\frac{1}{n}$, we have the following rule.

Multiply the numerator of the proposed fraction by n, and divide the product by the denominator.

Form, then, a new fraction, having for numerator the entire part of the quotient, and n for denominator.

GENERAL OBSERVATIONS ON FRACTIONS.

64. It results, obviously, from the nature of the methods established for the calculus of fractions, that the four fundamental operations performed upon them, to wit: addition, subtraction, multiplication, and division, are reduced always to the same operations performed on entire numbers.

Thus, for example, the addition and subtraction of fractions is brought back, by the reduction to common denominators, to the addition and subtraction of their numerators.

In the same manner, multiplication of fractions is effected by multiplying the numerators together, and the denominators. Division of fractions becomes multiplication, after inverting the divisor.

We conclude from this, that the principles established in Nos. 25 and 26, upon the multiplication of entire numbers, are equally applicable to fractions; that is to say, 1st, *to multiply a fraction by the product of several others, is the same thing as* *to multiply the first fraction successively by each one of the factors of the product:* 2d, *the product of two or more fractions is the same in whatever order we perform their multiplication.* In fine, we can apply to fractions all the propositions established in (40), concerning the changes which the product of a multiplication, or the quotient of a division undergo, when we cause one of the terms of the operation to undergo certain changes. We can multiply or divide both terms of any fractional expression whatever by the same number, without altering its value; and so on of other principles. We can deduce from the definition of multiple and submultiple, or divisor of a number, that there exist fractions which are multiples and submultiples of other fractions, in the sense that the division of the multiple fraction by the submultiple gives an entire quotient. Thus, the fractions, $\frac{12}{23}$, $\frac{8}{23}$, $\frac{6}{23}$, ... are multiples of $\frac{2}{23}$, since they contain the latter, 6, 4, 3 . . . . times, without remainder.

In general, every fraction has for divisors its half, its third, its fourth, &c.; whence, it follows, that the number of its divisors is infinite, which is not true of entire numbers, if the divisors are to be entire.

Two fractions can also have *common divisors;* thus, $\frac{35}{48}$, $\frac{7}{24}$, have for common divisor the fraction, $\frac{7}{48}$, and all its submultiples; for the quotients of $\frac{35}{48}$ and $\frac{7}{24}$, divided by $\frac{7}{48}$, are respectively 5 and 2, entire numbers. We can, then, generally establish, with relation to fractions, properties analogous to those which we have proved concerning the *greatest common divisor*, and the *least common multiple* of two or more numbers.

CHAPTER III.

COMPOUND NUMBERS.

65. The theory of *compound numbers* we place here as an immediate application of the theory of vulgar fractions. The units of smaller denominations being *fractions* of the principal units, or units of higher denominations, and fractions being really nothing more than units of lesser value than the principal unit with which they are compared. We can thus, in the number, $5\frac{4}{9}$, regard the ninths as simple units, and 5 as a number made up of compound units, each one equal to 9 times the simple unit; and the 9 under the 4 is the sign or denominator, showing the relative value of the simple units expressed by the number, 4. Thus, we have seen, in (No. 8), that in order to value quantities smaller than the *principal* unit, we conceive this unit divided into a certain number of equal parts, which we regard as forming new units. In the theory which now occupies our attention, the principal unit is first divided into a small number of equal parts, then these are divided into others, and these new parts into others, &c., &c.

Thus, for coin, the pound sterling, English, is divided into 20 parts, called shillings; the shillings into 12 parts, called pence, &c. In the same manner, the unit of length, the yard, is divided into 3 parts, called feet; the foot into 12 parts, called inches, &c.

66. Every art, each trade, each country, subdivides the principal unit, according to its own method.

The following tables give for the most important of these quantities, the principal units and their subdivisions; that is to say, those which follow the analogies of vulgar fractions. The decimal divisions of the principal units we reserve for the chapter on decimal fractions.

TABLES

In the estimation of time, the year is adopted as the principal unit; the subdivisions being, *months, weeks, days, hours, minutes, seconds.*

The year is divided into 365 days.
The day .... 24 hours.
The hour .... 60 minutes.
The minute............... 60 seconds.

(The minutes and seconds are generally indicated by ′ and ″.)

Or we may write the table thus:

One second = $\frac{1}{60}$ of a minute.
One minute = $\frac{1}{60}$ of an hour.
One hour = $\frac{1}{24}$ of a day.
One day = $\frac{1}{7}$ of a week.

Ex. — 5 days, 6 hours, 25 minutes, and 36 seconds, may be written either in columns, 5*d*, 6*h*, 25′, 36″, or thus:

$$5 + \frac{6}{24} + \frac{25}{60} \text{ of } \frac{1}{24} + \frac{36}{60} \text{ of } \frac{1}{60} \text{ of } \frac{1}{24}.$$

COINS.

Of coins, we give only the chief divisions of the English currency; the American and French coming under the decimal systems.

English Money.

One pound sterling = £ is divided into 20 shillings.
One shilling = *s.* .... 12 pence.
One penny = *d.* .... 4 farthings.

Or we may write it thus:

One farthing = $\frac{1}{4}$ of a penny.
One penny = $\frac{1}{12}$ of a shilling.
One shilling = $\frac{1}{20}$ of a pound.

Ex. — 5 pounds sterling, 6 shillings, and 10 pence, may be written £5, 6*s.*, 10*d.*, or £5 + $\frac{6}{20} + \frac{10}{12}$ of $\frac{1}{20}$.

WEIGHTS.

The standard avoirdupois pound of the United States, is the weight of 27·7015 cubic inches of distilled water weighed in air, at a fixed temperature. This gives us a fixed unit of comparison, or a principal unit of weight, of which the other divisions of the table are either multiples or submultiples.

TABLE OF AVOIRDUPOIS WEIGHT.

The ton	is divided into 20 hundreds	= cwt.
The hundred weight	4 quarters	= qrs.
The quarter	28 pounds	= lbs.
The pound	16 ounces	= oz.
The ounce	16 drams	= dr.

The *cwt.* in this table contains 112 *lbs.*, but the *cwt.* of one hundred pounds is very generally adopted in commerce, as more convenient, and much better adapted to the decimal system of the Federal money.

TROY WEIGHT.

The standard Troy pound of the United States is the weight of 22·794377 cubic inches of distilled water, weighed in air at a given temperature.

TABLE.

The pound (℔)	is divided into 12 ounces	= oz.
The ounce	20 pennyweights	= dwt.
The pennyweight	24 grains	= grs.

(7000 grains Troy make 1 *lb.* avoirdupois.)

The Apothecaries' weight for mixing medicines has the same principal unit as the Troy weight, but differs only in its subdivisions.

TABLE.

The pound (℔) is divided into	12 ounces	= ℥.
The ounce	8 drams	= ʒ.
The dram	3 scruples	= ℈.
The scruple	20 grains	= gr.

(The English pound, Avoirdupois and Troy, differ a little from those of the United States).

MEASURES OF LENGTH, AREA AND VOLUME.

Long Measure.

The principal unit of length is the yard, which is determined on the principle in physics that the pendulum which vibrates once in a second at the same place on the earth's surface, under the same surrounding circumstances, has a fixed and invariable length. This pendulum, or metal rod, is then divided off accurately, and a certain number of these subdivisions is called a yard. For the United States, the length of the pendulum is determined in New York city.

TABLE.

12 inches	make	1 foot.
3 feet		1 yard.
6 feet		1 fathom.
5½ yards		1 pole or perch.
40 poles		1 furlong.
8 furlongs		1 mile.
3 miles		1 league.

MEASURE OF AREA, OR SQUARE MEASURE.

The principal unit here, with which surfaces are compared, is a square whose side is 1 yard, or square yard.

TABLE.

144 square inches make .... = 1 sq. foot.
9 sq. feet .... = 1 sq. yard.
30¼ sq. yards .... = 1 sq. pole or *perch.*
40 perches .... = 1 rood.
4 roods .... = 1 acre.

The acre then contains 4840 sq. yards. For larger areas, we have the square, one of whose sides is a mile. This square mile contains 640 acres, (*called a section* in the public lands of the United States).

CUBIC, OR SOLID MEASURE.

The unit of volume, or solid measure, is a cube having one yard for its side, the other divisions being either multiples or subdivisions of this.

TABLE.

1728 cubic inches = 1 cubic foot.
27 cubic feet = 1 cubic yard, &c., &c.

The relations between the three tables of long measure, square and cubic measure, depend upon simple geometrical principles, which the student will find developed in any elementary work upon that subject.

LIQUID MEASURE.

The standard gallon of the United States is the *wine gallon*, which is equal to 231 cubic inches.

The gallon is divided into 4 quarts.
The quart .... 2 pints.
The pint .... 4 gills.

For the higher measures,

63 gallons = 1 hogshead.
2 hogsheads = 1 pipe or butt.
4 hogsheads = 1 tun.

DRY MEASURE.

The principal unit is the *bushel.* The standard bushel of the United States measures 2150·4 cubic inches. The names of the subdivisions, though the same as in liquid measure, do not represent the same volumes. The *gallons, quarts,* and *pints,* in *liquid measure,* measure respectively, 231, $57\frac{3}{4}$, and $28\frac{7}{8}$ cubic inches; while in *dry measure,* they measure $268\frac{4}{5}$, $67\frac{1}{5}$, and $33\frac{3}{5}$ cubic inches respectively.

TABLE.

The bushel is divided into 4 pecks.
The peck .... 2 gallons.
The gallon .... 4 quarts.
The quart .... 2 pints.

(The English imperial gallon measures 277·274 cubic inches.)

We see from these tables the great importance of determining accurately the standard of length, as all the other principal units of commerce depend upon this. Thus, the standard of *dry* and *liquid measure* is a certain number of cubic *inches.* The standard *weight* is a certain number of cubic *inches* of water. The standard of money is a coin containing a given *weight* of metal.

67. We call a *compound number* every concrete or denominate number, which contains, at the same time, one or more principal units of a certain species, and one or more subdivisions of this unit, or simply one or more subdivisions of the principal unit alone. Thus, £10 12*s.* 8*d.*, 25 *mls.* 4 *fur.* 7 *yds.*, 70 *days,* 23 *hours,* 10 *min.*, or simply 12*s.* 10*d.*, 4 *h.* 10 *min.*, &c., &c., are compound numbers.

But, £10, or 10*s.*, or 23 hours, are not compound numbers, considered thus isolated. The resolution of the following questions serves as a basis for the four fundamental operations on compound numbers.

68. *Question first. — A compound number being given, to reduce this number, or express it in units of the smallest subdivision of the principal unit.*

Given, for example, 2 *lb.* 4 *oz.* 17 *dwts.* 5 *grs.*, to be converted into grains.

It results from the tables, that the pound equals 12 ounces.

Therefore, 2 ℔. 4 *oz.* $= 2 \times 12 + 4 = 28$ *oz.*, or, $2\frac{4}{12}$ ℔ $= \frac{28}{12}$ ℔. In the same manner, the ounce equals 20 dwt. Hence, 28 *oz.*, 17 *dwt.* $= 28 \times 20 + 17$ pennyweights $= 577$ *dwt.*, or, $28\frac{17}{20} = \frac{577}{20}$ *oz.* Again, 577 *dwt.* 5 *grs.* $= 577 \times 24 + 5$ *grs.* $= 13853$ grains, or, $577\frac{5}{24} = \frac{13853}{24}$ *dwt.*

GENERAL RULE. — *Multiply, first, the number of principal units by the number of units of the first subdivision which the principal unit contains, and add to the product the units of this first division, which are contained in the given number. Then multiply the result thus obtained by the number of units of the second subdivision which the first contains, and add to this second product the units of the second subdivision, which enter into the given compound number; and thus, in succession, until we arrive at the last subdivision or denomination.*

We will find, by this method,

1st. — 59 ℔. 13 dwts. 5 gr. = 340157 gr.
2d. — 121 ℔. 0*s.* $9\frac{1}{2}$*d.* = 58099 halfpence.
3d. — 23 *h.* 55 *min.* 19″ = 26119 seconds.

69. *Second question. — Reciprocally, given a number of units of a certain division of the principal unit, to be converted into a compound number.* The rule to be followed is evident from what precedes, and can be enunciated thus:

Divide, first, the proposed number, by the number which expresses how many times the given subdivision is contained in the subdivision next higher; we obtain thus for quotient, a certain number of units of this next higher division, and for remainder the units of the given denomination which are to enter into the compound number sought. Divide, then, the quotient obtained *by the number which expresses how many times the subdivision next higher is contained in the denomination higher by two than the given one; we obtain a new quotient, which contains a certain number of units of the third denomination, of which we have just spoken, and a new remainder, expressing the units of the denomination next to the given one, which make part of the compound number sought. Continue thus, until the quotients cease to be divisible by the number expressing the relation between the value of two successive denominations.*

N. B. If we obtain 0 for any one of the remainders, this proves that the denomination corresponding is wanting in the number sought.

Let us apply this rule to the first example of (68).

$$
\begin{array}{llll}
13853 & |\ 24 & & \\
\underline{120} & & & \\
 & 577 & |\ 20 & \\
\underline{185} & & & \\
 & \underline{177} & |\ \underline{28} & |\ \underline{12} \\
\underline{173} & & & \\
 & 17 & |\ 4 & |\ 2 \\
5 & & &
\end{array}
$$

Result, 2 ℔. 4 *oz.* 17 *dwt.* 5 *gr.*

Or thus:

$$
\begin{array}{lll}
13853\,gr. \div 24 = & 577\,dwt. & +\ 5\,gr. \\
577\,dwt. \div 20 = & 28\,oz. & +\ 17\,dwt. \\
28\,oz. \div 12 = & 2\text{ ℔} & +\ 4\,oz.
\end{array}
$$

70. *Question third.* — *To convert a given compound number into a fraction of the principal unit.*

This is also a consequence of (68).

Take, for example, 2 ℔. 4 *oz.* 17 *dwt.* 5 *gr.* This, reduced to grains, gives 13853 grs.; and, by the tables of (66), 1 gr. is $\frac{1}{20}$ of $\frac{1}{24}$ of $\frac{1}{12}$ of a pound, or $\frac{1}{1440}$ of a pound; the required fraction is obviously then $\frac{13853}{1440}$ of a pound.

RULE. — *Commence by reducing the given compound number into units of the lowest denomination which it contains; then form a fractional number which has for numerator the number thus obtained, and for denominator the number of units of this lowest denomination which the principal unit contains.*

We will find, by this method, 23 *h.* 55′ 9″$=\frac{86119}{60\times 60}=\frac{86119}{3600}$ of an hour.

71. *Question fourth.* — *Reciprocally, given, any fractional number of the principal unit of a certain denomination, to convert it into a compound number.*

Given, for example, $\frac{5}{7}$ of a mile to be converted into furlongs, poles, &c. Since each mile equals eight furlongs, $\frac{5}{7}$ of a mile is $\frac{5}{7}$ of 8 furlongs; equals $\frac{40}{7}$ of 1 furlong. We then divide 40 by 7; the quotient, 5, expresses obviously the furlongs, and the remainder, 5, with the divisor, 7, for denominator, is a fraction of a furlong which it is necessary to reduce to poles. Now, 1 furlong equals 40 poles; hence, $\frac{5}{7}$ of a fur. $=\frac{5}{7}$ of 40 poles, equals $\frac{5\times 40}{7}$ of 1 pole. Performing the operations here indicated, we have 28 for quotient, and $\frac{4}{7}$ of a pole, for the fraction corresponding to the remainder; $\frac{4}{7}$ of a pole $=\frac{4}{7}$ of $5\frac{1}{2}$ yards $=\frac{22}{7}$ of a yard, which is equal to $3\frac{1}{7}$. Hence, the required compound number is 5 fur. 28 pls. $3\frac{1}{7}$ yds.

OPERATION.

$$\begin{array}{l|l} 5 & \\ 8 & \\ \hline 40 & 7 \\ \hline 5 & 5 \\ 40 & \\ \hline 200 & 7 \\ \hline 60 & 28 \\ \hline 4 & \\ 5\frac{1}{2} & \\ \hline 22 & 7 \\ \hline 1 & 3 \end{array}$$

GENERAL RULE. — *In order to convert a fractional number of any principal unit into a compound number, obtain first the entire number, if there be one, contained in the fraction; you obtain thus a certain number of the principal units.*

Multiply, then, the remainder of this division, by the number which expresses how often the principal unit contains the next lower subdivision, and divide this product by the denominator of the given number; we thus obtain a certain number of units *of this next lower subdivision, and a second remainder. Proceed with this remainder in the same manner, until you arrive at a result with no remainder, or exhaust the subdivisions of the principal unit.*

N. B. Principal unit can apply to any one of the denominations of the tables (66), which has itself been subdivided; that is to say, every subdivision can be principal unit to the subdivisions below it.

72. *Remark I.*—The operations of the two last rules can serve as verification for each other. Thus, in applying (71), to the fractional numbers of (70), we ought to reproduce the four compound numbers which correspond. In the same manner, we can verify the result of (71), by means of the rule in (70).

73. *Remark II.*—The principles which have just been developed would be, properly speaking, sufficient to permit us to perform the four fundamental operations of arithmetic upon compound numbers.

We would thus pursue the following method:

1st. *Transform the compound numbers, each one into a fraction of the principal unit corresponding.*

2d. *Perform upon these fractional numbers the operation proposed (according to the rules of the calculus of fractions) which will give for a result a fractional number.*

3d. *Convert this fractional number into a compound number of the species indicated by the nature of the question.*

Nevertheless, since the direct methods of performing the four fundamental operations upon compound numbers give rise to important observations, and offer for the theories which we shall develop later, useful applications, we will proceed to discuss them, as simply as possible, with very simple examples.

ADDITION AND SUBTRACTION.

74.—1st. *Addition.*—*Place* (as in abstract entire numbers), *the given numbers, one under another, so that the units of the same denomination fall under each other; after which, make* *the addition of the units contained in each column, commencing on the right.*

If the sum of the units contained in a column exceeds the number which expresses how many times the unit of the denomination corresponding is contained in the unit of the next higher denomination, we divide the sum obtained by this number (69); *we obtain thus a remainder, (possibly* 0,) *which we write below the horizontal line drawn under all the columns, and a quotient which we carry to the units of the following column; we operate in the same manner upon this column, and upon each successive column.* (This rule is obviously established by the same reasoning which was given for the rule in simple addition of abstract numbers).

2d. — *Subtraction. — Write the smaller number under the greater, so that the units of the same denomination fall under each other; then subtract successively, one from the other, the units of each denomination, commencing with the lowest.*

When, in any one of the columns, the number of units to be subtracted is greater than the number from which it is to be taken, we add to this latter (14), *a unit of the denomination next higher, converted into units of the denomination on which we are operating; the partial subtraction becomes possible. We must take care, however, to augment the next number to be subtracted by the one unit borrowed from this denomination.*

(This rule is obviously founded on the reasoning for subtraction of simple numbers).

We give below some examples:

£	*s.*	*d.*	
17	13	4	
13	10	2	
10	17	3	
8	8	7	
3	3	4	
	8	8	
54	1	4	
23	32	0	*Proof.*

℔.	*oz.*	*dwt.*	*gr.*	
14	10	13	20	
13	10	18	21	
14	10	10	10	
1	4	4	4	
45	0	7	7	
13	2	12	0	*Proof.*

SUBTRACTION.

£.	s.	d.	
57	13	8	
49	17	11	
7	15	9	
57	13	8	*Proof.*

mls.	*fur.*	*pol.*	*yd.*	*ft.*	*in.*	
14	3	17	1	2	1	
10	7	30	2		10	
3	3	26	$4\frac{1}{2}$	1	3	
14	3	17	1	2	1	*Proof.*

The methods of verification are the same as in abstract numbers, taking care to preserve the relative values of the units carried from the columns of higher denominations to lower, as in the verification of addition, and from lower to higher, as in subtraction.

MULTIPLICATION.

75. To multiply a compound number by a simple factor, we consider the multiplication of each denomination of the compound number as a separate question; then reduce the partial products to compound numbers by (69), and add these compound numbers by last article. Or, what is the same thing, *commence on the right hand, and proceed with the multiplication as in simple numbers, taking care to preserve the proper relative values between the successive columns.*

Thus, £4 13*s.* 3*d.* to be multiplied by 9.

£	*s.*	*d.*
4	13	3
		9
41	19	3

9 times 3 gives 27, which we reduce to shillings, giving 2 for quotient, with 3 remainder; set down the 3, and carry the 2 to the next multiplication; 9 times 13 gives 119 = £5 19*s.* We set down the remainder, and add the 5 to the product of 9 by £4, giving £41. If we have one denominate number to be multiplied by another, we reduce multiplicand and multiplier to fractional numbers of their principal units by (70); then multiply, and reduce the result back to the compound number required by the

question; or we may simply reduce the multiplier to such a fraction, and proceed as in the first example.

Example. — £2 5*s.* to be multiplied by 10℔ 5*oz.* avoirdupois. We may either multiply $\frac{45}{20}$ by $\frac{165}{16}$, and reduce the result to pounds and shillings, or we may multiply £2 5*s.* by $\frac{165}{16}$, reducing each result separately.

76. *Remark.* — It results obviously from this mode of proceeding,

1st. That although the multiplier is a denominate number, yet we consider the principal unit of this factor and its subdivisions as abstract numbers, which express the number of times we must take the multiplicand, and what parts of it we must take, in order to obtain the required result; but we preserve always in the multiplicand its essential quality of concrete number.

2d. That all the partial products and the total product are always of the same nature as the multiplicand.

Certain questions of Geometry, however, namely: those which have for their object the measure of surfaces and volumes, give rise to operations which form exceptions to this general principle. The considerations on which these are founded do not belong to arithmetic.

DIVISION.

We will dwell but little on this operation, in effecting which, in general, it is better to apply the method established in (73). Nevertheless, we will consider the two principal cases which can present themselves.

77. *Case I.* — *In which the dividend and divisor are compound numbers of the same species.* For example: — Required, How many yards of a certain work can we have executed for £75 19*s.* 5*d.*, if one yard cost £8 15*s.* 6*d.*?

It is clear that, for the resolution of this question, we must determine how many times the smaller of these two compound numbers is contained in the greater. This is effected, 1st, by reducing the two numbers to the lowest denomination which enter them; 2d, by then dividing the entire numbers thus obtained one by the other. The quotient is at first an abstract number, which, according to the enunciation of the question, can then be expressed in yards, feet, inches, &c. Converting the two given numbers to pence, we find 18,233*d.* and 2106*d.* The fractional number then will be $\frac{18233}{2106}$, which can be converted into yards, &c., by rule in Art. (71).

78. *Case II.* — *That in which the dividend and divisor are of different species.* In this case, whatever be the question proposed, the quotient must express principal units of the same species as the dividend; since it is necessary that the dividend, considered as a product, must be of the same species as one of its factors. But then, the compound divisor, being converted into a fractional number of the principal unit, becomes an abstract number, by which we must divide the dividend, which is done by multiplying the dividend by this fraction inverted (60).

79. *Remark I.* — We conclude from the above, 1st. That in every division of compound numbers, if the two numbers are of the same species, the quotient is considered first as an abstract number, which we make express the units and subdivisions of units, fixed by the enunciation of the question. This quotient is to be the multiplier in the verification of the operation by multiplication.

2d. That if, on the contrary, the two terms of the division are of different species, the quotient expresses necessarily units of the same species as the dividend; while the divisor, though compound at first, is to be regarded as an abstract number, which plays the part of multiplier in the verification of the operation.

80. *Remark II.* — So far, we have only given one method of verifying multiplication, viz: the method by division, and reciprocally. But in the practice of the operations upon compound numbers, it is generally more convenient to verify, 1st, Multiplication, by doubling one of the two factors, and taking the half

of the other; then performing the operation anew with the resulting numbers. 2d, Division, by doubling the two terms of the division. We avoid thus the difficulties arising from the vulgar fractions, which ordinarily accompany the results arrived at.

It is evident that this means of verification can also be employed with entire abstract numbers.

EXERCISES.

1. Find a number, the $\frac{1}{2}$, the $\frac{1}{4}$, $\frac{2}{5}$, and $\frac{3}{4}$ of which, added together, form a sum which, diminished by 139, gives 1289 for remainder.

2. A reservoir is filled by four different pipes. The first can fill it alone in 5 hours; the second in 7 hours; the third in 9 hours; the fourth in 11 hours. Required, the time of filling the reservoir, all four pipes being opened at once.

3. The population of Asia is estimated at 390,257,000 inhabitants: Required the population of Europe, Africa and America; knowing that the population of Europe is $\frac{7}{13}$ of the population of Asia; that of Africa, $\frac{3}{11}$ of that of Europe; and that of America, $\frac{1}{11}$ of the same.

4. The sea covers $\frac{11}{14}$ of the whole surface of the globe. The surface of Asia is equal to $\frac{121}{27}$ of that of Europe; that of Africa is $\frac{22}{7}$ of the same; that of America, $\frac{111}{29}$; and that of Oceanica, $\frac{31}{47}$; we know, besides, that Africa has a superficies of 13,450,000 square miles. Calculate the superficies of the other parts of the world, and deduce the number of square miles in the whole surface.

5. Demonstrate that, by adding the same number to the two terms of a fractional number, we obtain a result which approaches unity more as the number added is greater. Show that the difference between the result and unity can become less than any given quantity.

6. Find the method of obtaining the greatest common divisor of two or more fractions. Apply to the fractions,

$$\frac{3}{4}, \quad \frac{5}{6}, \quad \frac{7}{18}, \quad \frac{19}{24}.$$

7. Demonstrate the method for obtaining the least common multiple of several fractions.

Apply to the fractions, $\frac{5}{12}$, $\frac{11}{18}$, $\frac{19}{60}$.

8. What is the greatest common multiple of the fractions, $\frac{28}{57}$, $\frac{4}{21}$, and $\frac{18}{19}$, less than 100,000.

9. What will be the price of a piece of stuff, $23\frac{7}{24}$ yards long, each yard costing £5 10*s.* 6*d.*?

10. 87 ℔. 10 *oz.* 5 *dr.*, of a certain material, was bought for 50£ 11*s.* 9*d.* What is the price per pound?

CHAPTER IV.

Of Decimal Fractions, and their Principal Properties — Of the Decimal Systems of Compound Numbers.

I. — DECIMAL FRACTIONS.

81. *Introduction.* — In the ordinary system of numeration, the most simple method, and the most convenient one of subdividing unity, *is the subdivision into successive parts, decreasing in a ten-fold ratio.* From this mode of subdivision result *fractions which have for denominators unity, followed by one or more zeros*, and these fractions we call *decimal fractions.*

This mode of subdividing unity offers great advantages, inasmuch as it reduces immediately, or at least by very simple transformations, all the operations upon fractional numbers, to simple operations upon entire numbers. These methods we will develop after having made known the numeration of decimal fractions;

that is, their nomenclature, and the manner of writing them in figures.

82. *Numeration of Decimals.* — As, by increasing unity tenfold, one hundred-fold, &c., successively, we form new units, to which we give the name of tens, hundreds, thousands, and so-forth, in the same manner we conceive unity to be divided into 10 equal parts, which we call *tenths*, each tenth divided into 10 equal parts, which we call *hundredths*, (because the principal unit contains 10 times 10, or 100 of these new parts or units); then each hundredth divided into 10 equal parts, called thousandths, and so on; thus giving ten thousandths, hundred thousandths, &c.

In the second place, it results, (5), from the fundamental principle of the written numeration of entire numbers, that the figures, proceeding from right to left, have their relative value increased tenfold for each place to the left, and decreased tenfold, going from left to right. Whence it follows, that if to the right of an entire number written in figures we place new figures, taking care always to distinguish them by any sign whatever, a comma or point for example, from the entire number, we shall thus represent successive parts of unity, decreasing tenfold to the right; that is, tenths, hundredths, thousandths, &c.

Thus, the collection of figures, 24,75, expresses 24 *units*, 7 *tenths*, and 5 *hundredths;* 5,478 equals 5 *units*, 4 *tenths*, 7 *hundredths*, and 8 *thousandths*.

83. Let it be required to enunciate in ordinary language the number 56,3506. This number can at first be enunciated 56 *units*, 3 *tenths*, 5 *hundredths*, 0 *thousandths*, and 6 *ten thousandths*. But 3 tenths are equal to 30 *hundredths*, or 300 *thousandths*, or 3000 *ten thousandths;* in the same manner, 5 *hundredths* are equal to 50 *thousandths*, or 500 *ten thousandths* The number can then be enunciated 56 units, and 3506 ten thousandths.

Thus, in order to enunciate in ordinary language a decimal fractional number written in figures, *we must enunciate separately the entire part, and then enunciate the part which is to* *the right of the comma, as an entire number, giving at the close the name of the unit of the last decimal subdivision.*

Thus, 7,49305 represents 7 *units* and 49305 *hundred thousandths.* In the same manner, 249,007,056 represents 249 units and 7056 millionths. We can also, if we wish, include in one single enunciation the entire as well as the decimal part

Take, for example, the number 56,3506. As one unit equals 10 tenths, or 100 hundredths, 1000 thousandths, &c., it follows, that 56 units are equal to 560000 ten thousandths; and, consequently, 56,3506 represents 563506 ten thousandths. That is, we must, *after enunciating the number as if it had no comma,* place at the end of the number thus enunciated, the *name of the last subdivision.* It is customary, however, to enunciate the entire part separately.

We will indicate a method for enunciating the decimal part, which, in general, is more convenient in practice. After announcing the entire part, as we have just said, *separate* mentally the *decimal part into periods of three figures, beginning at the comma,* (the last period having often only one or two figures); *enunciate then each* period or division separately, and place at the end of each partial enunciation the name of the last unit of the period.

Example.—The number, 2,74986329, is enunciated; 2 units, 749 thousandths, 863 millionths, 29 hundred millionths.

84. Reciprocally, we propose *to write in figures a decimal fraction enunciated in ordinary language.*

Required to write in figures the number; *twenty-nine units, three hundred and fifty-four thousandths.* Write first the entire part, 29; then, as 300 thousandths are equal to 3 tenths, and 50 thousandths equal 5 hundredths, place a comma to the right of 29, and write successively the numbers 3, 5, and 4; we thus have 29,354.

In like manner, *one hundred and nine units, two thousand and three ten thousandths,* are written 109,2003.

Required, again, to write the number *eight* units, *thirty-seven* thousandths. As thirty thousandths make 3 hundredths, and as there are no *tenths* in the number enunciated, we write 8,037; that is to say, we make the same use of the 0 in both these last cases as in whole numbers, placing it here to the right of the comma, to take the place of the tens which are wanting, and to give the figures which follow their true value.

General Rule. — In order to write, in figures, a decimal enunciated in ordinary language, *commence by writing the entire part, and after it a comma or point; then write successively, to the right of this point, the figures which represent the tenths, hundredths, &c., included in the number, taking care to replace by zeros the different orders of units which are wantiug.* If there is no *entire part, write a* 0 *to take the place of it, and proceed with the decimal part as before.*

Thus, seventeen hundredths are represented by 0,17; one hundred and twenty-five *ten thousandths* by 0,0125.

It may happen that, in the enunciation of the number, the entire part is not distinguished from the decimal part. We must then write the number as if it expressed entire units, and then place a point so that the last figure to the right shall express the units of the last subdivision of the number enunciated.

For example, in order to write the number four thousand, two hundred and fourteen *hundredths,* write first 4214; and, as the last figure must express *hundredths,* place the comma between the 2 and 1, giving 42,14. Two hundred and fifty-three thousand and twenty-nine ten thousandths, are represented by 25,3029.

85. *Decimal fractions placed under the form of vulgar fractions.* A fraction being composed of two terms, the numerator and the denominator, the comma serves, in the method which we have just developed, to indicate the denominator, which is equal to unity, followed by as many zeros as there are decimal figures; that is, figures to the right of the comma. The numerator, we

have seen, is composed of the collection of figures to the right of the comma. Or, if we consider the entire part as reduced to a fraction, the numerator is then the number given, with the comma stricken out. Thus, the number, 23,5037, put under the form of a vulgar fraction, is $23\frac{5037}{10000}$, or, $\frac{235037}{10000}$. The number, 2,00409, is equal to $2\frac{409}{100000}$, or, $\frac{200409}{100000}$. Finally, 0,0002154, is equal to $\frac{2154}{10000000}$. Reciprocally, $2\frac{53}{1000}$, or, $\frac{2053}{1000}$, is equal to 2,053; $\frac{172049}{10000}$ is equal to 17,2049.

These two transformations are of continual use in the calculus of decimal fractions.

86. *Changing the place of the point.*—If, in a decimal fraction, we advance the point one or more places to the right, we multiply the number by 10, 100, 1000, &c.; and if, on the contrary, we place it one or more places farther to the left, we divide the number by 10, 100, 1000, &c.

For, let the number be 153·07295.

Suppose we advance the point three places to the right, which gives 153072·95. The two numbers are now $\frac{15307295}{100000}$, and $\frac{15307295}{100}$. Now, the denominator of the second number is 1000 times smaller than that of the first, while the numerator is the same. Then, the second fraction is 1000 times greater than the first. On the contrary, remove the point two places towards the left, it becomes 1·5307295, or, $\frac{15307295}{10000000}$, a fraction evidently 100 times smaller than the given one. We could establish the same thing by reasoning thus:—By changing the place of the point, the value of each figure becomes 10, 100, 1000, &c., times greater or smaller. Thus, in comparing 153072·95, with 153·07295, we see that the figure 3, which expresses in the latter *simple units*, expresses now *thousands;* the figure 5, to the left of the figure 3, which expressed *tens*, represents now *tens of thousands;* and the same with the other figures.

87. *Zeros placed to the right of a decimal fraction.*

By annexing any number whatever of zeros to the right of a decimal fraction, we do not change its value.

Thus, 3·415 is equivalent to 3·4150, 3·41500 . . . .; for these numbers can be (85), put under the form,

$$\frac{3415}{1000}, \frac{34150}{10000}, \frac{341500}{100000}, \ldots.;$$

Now, the last two fractions are nothing more than the first, with its two terms, multiplied by 10, 100, which (43), does not change its value. Then, &c. . . . .

Or, we may observe that zeros, placed to the right of decimal figures already written, do not change *their* value; and, as these zeros have no value of themselves, the fraction remains always the same. As the value of a figure in a decimal fraction depends entirely on the number of places it is distant from the point, it is obvious that we do alter this value by prefixing zeros between the decimal point and the first decimal figure.

88. *Reduction of several decimal fractions to the same denominator.*

The principle which has just been established, gives us a method of reducing several decimal fractions *to the same number of decimal figures*, without changing their value; or, in other terms, *to the same denominator*.

For example, the fractions

12·407 | 0·25 | 7·0456 | 23·4
are equal to 12·4070 | 0·2500 | 7·0456 | 23·4000.

They have 10000 for common denominator. These preliminary ideas being established, we pass to the four fundamental operations upon decimal fractions.

ADDITION AND SUBTRACTION.

89. *We perform the addition of decimal fractions in the same manner as we do that of entire numbers, after reducing them all to the same denominator, and we point off in the result as many decimal places as there are in any one of the reduced numbers, or the greatest number which any one of the given numbers contains.*

A single example will suffice to illustrate and make plain this rule.

Given, to add the numbers

32·4056 | 245·379 | 12·0476 | 9·38 | and 459·2375.

$$\begin{array}{rl} 32{\cdot}4056 & \\ 245{\cdot}3790 & \\ 12{\cdot}0476 & \\ 9{\cdot}3800 & \\ 459{\cdot}2375 & \\ \hline 758{\cdot}4497 & \\ \hline 121{\cdot}2210 & \textit{Verification.} \end{array}$$

We write, first, *one* zero to the right of the second number, and *two* to the right of the fourth; we then place the numbers thus prepared, one under another, so that the units of the same order correspond, and then make the addition in the ordinary manner. We find for result, 7584497; or, separating the four figures to the right, 758·4497; because the numbers added express units of the order of ten thousandths.

In practice, we can dispense with writing the zeros to the right of the numbers, which contain fewer decimal places than the others, provided we take care to arrange the units of the same order in the same column.

Subtraction is performed in the same manner as in entire numbers, after we have reduced the decimals to the same denominator (88).

Example. — Given, to subtract 23·0784 from 62·09.

$$\begin{array}{rl} 62{\cdot}0900 & \\ 23{\cdot}0784 & \\ \hline 39{\cdot}0116 & \\ \hline 62{\cdot}0900 & \textit{Verification.} \end{array}$$

We write two zeros to the right of the 62·09, which gives 62·0900; we then perform subtraction in the usual manner, taking care to separate four decimal figures to the right of the result.

These methods are obviously founded upon the fact that the units of different orders, in decimal fractions, having the same relations of magnitude, one to the other, as in entire numbers, we have the same operations to be performed with the figures to be *carried* as in entire numbers.

MULTIPLICATION.

90. In order to perform this operation, *multiply the two given numbers one by the other, without regarding the comma or point which they contain; then separate by a point, from the right of the product thus obtained, as many decimal figures as there are in both factors.*

Required, for example, to multiply 35·407 by 12·54. We find first for the product of the two numbers, the points being disregarded, 44400378. Pointing off, then, on the right of the product, 3 + 2, or 5 figures, we obtain for the required product, 444·00378. In order to see the reason of this method, we remark, that the two given numbers are equal to (160), $\frac{35407}{1000}$, and $\frac{1254}{100}$. Whence we deduce the product by the rule in (57), $\frac{35407 \times 1254}{1000 \times 100}$; that is to say, it is necessary, 1st, to multiply the two numbers, disregarding the point; 2d, to divide this product by 100000, or unity, followed by as many zeros as there are decimal figures in the two factors, which is equivalent to separating 5 decimal figures on the right of the product. The method is thus justified. Or, we may reason thus: by removing the point from the multiplicand, we multiply it by 1000; since, at first, it expresses thousandths, but after the multiplication, principally units; then, the product is 1000 times too great. In the same manner, by removing the point from the multiplier, we render it 100 times greater. Thus, by the suppression of both points, the product is rendered 100000 times too great; then, in order to bring it back to its just value, it must be divided by 100000, or five figures must be pointed off for decimals on the right. The reasoning would obviously be the same, whatever be the number of decimals in the two factors. It can happen that one of the two numbers only contains decimals. In this case, *we point off, on the right of the product, as many decimal figures as there are in this number.* The demonstration is too easy and obvious to detain us.

We will find, according to these rules,

1st. The product of 4·057 by 9·503, is 38·553671.

2d. The product of 4·0015 by 29, is 116·0435.

3d. The product of 0·03054 by 0·023, is 0·00070242.

N. B. This last example deserves some attention. Suppressing the point in the two factors, and performing the multiplication, we find for a product, 70242; but, as there are *five* decimals in the multiplicand, and *three* in the multiplier, there must be *eight* of them in a product which contains only *five* figures. In order to remove the difficulty, we observe, that as the product ought to express units of the 8th order of decimals, it suffices to write, on the left of 70242, zeros in such number that, the point being placed on the left of them, the last figure to the right shall occupy the 8th decimal rank. We write *three* zeros then on the left, besides one for the entire number, and obtain 0·00070242.

DIVISION.

91. Two principal cases present themselves. Either the dividend and divisor have the same number of decimals, or this number is different. In the first case, *suppress the point in the dividend and in the divisor; then operate upon the entire numbers which result from it. according to the ordinary rule of division.*

In the second, *commence by reducing the two given numbers to the same number of decimal places, or to the same denominator.* The second case thus becomes the first.

First Case.—Required to divide 47·359 by 8·234. These two numbers can be put under the forms (85), $\frac{47359}{1000}$, $\frac{8324}{1000}$. Dividing them one by the other, according to rule for the division of fractions (59), we have $\frac{47359}{1000} \times \frac{1000}{8234} = \frac{47359 \times 1000}{8234 \times 1000} = \frac{47359}{8234}$, suppressing the factor, 1000, common to the two terms.

We see, then, that the quotient required is equal to that of the two given numbers with the point removed; and the rule above is proved. We can also say, the two decimal fractions having the same denominator, if we suppress the point, we multiply the two terms of the division by the same number, 1000; then, the value of the quotient remains the same. The division of 47359 by 8234, gives for the entire part of the quotient, 5, and for remainder, 6189; thus the total quotient is, $5\frac{6189}{8234}$.

92. *Valuation of the quotient in decimals.*—The vulgar fraction, which accompanies the entire part of the quotient, having terms pretty large, it is difficult to value it in its present state; moreover, it is natural to endeavour to express it in parts of the same species as the given numbers. We arrive at this now by the rule in (63):

$$\begin{array}{l|l}
47359 & 8234 \\
\hline
61890 & 5{\cdot}7516395\ \ldots\ . \\
\quad 42520 & \\
\quad\quad 13500 & \\
\quad\quad\quad 52660 & \\
\quad\quad\quad\quad 32560 & \\
\quad\quad\quad\quad\quad 78580 & \\
\quad\quad\quad\quad\quad\quad 44740 & \\
\quad\quad\quad\quad\quad\quad\quad 3570 & \\
\quad\quad\quad\quad\quad\quad\quad \ldots . &
\end{array}$$

After obtaining the entire part, 5, of the quotient, in order to make the remainder, 6189, express tenths, we multiply it (63), by 10; this we effect by placing a 0 on its right; then we divide 61890 by 8234; the quotient, 7, expresses then *tenths;* and we write it to the right of the figure 5, with a point before it. To the right of the new remainder, 4252, we place a 0, in order to convert it into *hundredths;* we then divide 42520 by 8, which gives the quotient, 5; this we place on the right of 7, and annex another 0 to the remainder, 1350; and so on, until we have obtained the number of decimal places which the enunciation question giving rise to the decision demands.

General Rule. — *In order to express, in decimals, the quotient of the division of two decimal numbers of the same denominator,* or (which is the same thing after the suppression of the point), *of any two entire numbers whatever,*

Commence by determining the entire part of the quotient, (which can be 0), *and write a point after it.*

Annex a zero to the right of the remainder; divide the number thus formed by the divisor; then place the quotient on the right of the point. Annex a 0 *to the right of the new remainder, and perform the division by the same divisor; write the quotient on the right of the two first. Continue thus until you have the number of decimals required.*

93. *Remark on these approximations.* — In the preceding example, we have carried the operation as far as the seventh decimal figure, in order to establish some principles upon the different degrees of approximation which can be obtained by the development of a number into decimals.

By taking at first only the two first decimal figures, we have 5·75 for the value of the quotient, *to within less than* 0·01, since the part neglected is obviously less than the unit of this order of decimals. Again, as this neglected part is less than 0·002, or $\frac{2}{1000}$, or $\frac{1}{500}$, it follows, that 5·75 expresses the value of the quotient to within less than $\frac{1}{500}$.

Now, if we take the three first decimal figures, we have 5·751 for the value of the quotient, *to within less than* 0·001, since the part which we neglect, 0·00063 . . . . is less than 0·001. But here we must make an important observation. As the figure 6 exceeds 5, it follows that 0·0006 exceeds 0·0005, or *a half unit of the order thousandths;* then, by taking 5·752, instead of 5·751, for the value of the quotient, we commit an error in thus taking more than the true value, less than is committed when we take 5·751 for this value; and we can say that 5·752 expresses the quotient, not only *to within less than* 0·001, but to within less than *the half of* 0·001.

Generally, *whenever the figure which follows that one at which we wish to stop in the division, is less than* 5, *we preserve the figure obtained, and we then have the value of the quotient to within less than a half unit of the denomination at which we stop. If, on the contrary, the figure which follows is equal or greater than* 5, *it is best to increase by one unit the last figure obtained, in order to obtain a value nearer the quotient; the error committed is an error of excess, but it is less than a half unit of the order at which we stop the operation.*

Thus, in the example above, we have successively for the quotient of the proposed division, 5·752, too great by less than a *half thousandth;* 5·7516, too small by less than a half ten thousandth; 5·75164, too great by less than a half hundred thousandth; 5,751640, too great by less than a half millionth. We will add, that when we have arrived at any decimal figure whatever, in the operation performed, the last remainder obtained shows whether the following figure of the quotient is greater or less than 5, without necessarily calculating this figure.

If the remainder is less than half the divisor, the following figure of the quotient will necessarily be less than 5.

If this remainder is equal to, or greater than the half of the divisor, the next figure of the quotient will be equal to, or greater than 5.

Thus, in the example which we have just discussed, the eighth figure of the quotient must be less than 5; for, the remainder at which we stopped, 3570, is obviously less than the half of the divisor, 8234.

We have here given the whole theory of approximations in the valuation of fractional numbers in decimals.

94. *Case Second.*—This divides itself into two others:

Firstly,—The dividend *contains fewer decimal figures than the divisor.* We write on the right of the dividend the number of zeros necessary to reduce the two terms of the division to the same number of decimal places; and the question is solved by Case First without farther modification.

For example:—Required, to divide 2·405 by 0·03497. Placing two zeros to the right of the dividend, which gives 2·40500; then, suppressing the comma in both numbers, we perform the division of the two resulting numbers, 240500, and 3497, according to the rules in (91 and 92). We find thus the value of the quotient to within less than ·0001, to be 68·7732.

Secondly,—*The dividend has more decimal figures than the divisor; we can then employ two methods.*

1st. *Required to divide* 3·470456 *by* 1·027. If we suppress the point in the divisor, thus rendering it 1000 times as great, and if we advance the point in the dividend three places to the right, rendering it thus also 1000 times as great as at first, the quotient of the division of these two numbers resulting, will be the same as that of the given numbers. The question is thus reduced to dividing 3470·456 by 1027.

3470·456	1027
3894	3·379217
8135	
9466	
2230	
1760	
7330	
141	

After finding the entire part, 3, of the quotient, and the remainder, 389, instead of placing, as in (92), a zero to the right of this remainder, we bring down the figure 4, which expresses *tenths*, and perform the division, obtaining for quotient, 3, which we place on the right of the first, separating them by a point; we then bring down to the remainder, 813, the figure 5, which expresses *hundredths;* and we continue thus, until we have brought down all the decimal figures which are contained in the dividend. When we reach the remainder, 223, we place a zero on the right of it, and operate as in case first. We see that this method consists in suppressing the point in the divisor, taking care to remove it in the dividend as many places to the right as there are decimals in the divisor; then, in operating upon the resulting numbers, as in the first case, with this difference, that instead of annexing at first zeros to the right of the different remainders, we commence by bringing down successively all the decimal figures of the dividend.

2d. We take the same example, and commence by writing to the right of the divisor three zeros; that is to say, the number of zeros necessary to reduce the two terms to the same number of decimal places.

We have then to divide 3470456 by 1027000.

```
347056 | 1027(000
------ | --------
389456 | 3·379217
------
 81356
 -----
  9466
  ----
   2230
   ----
    1760
    ----
     7330
     ----
      141
```

In order to determine the entire part of the quotient, we commence by applying the rule of (38), for the division of entire numbers, when the divisor is terminated by zeros. We obtain thus the quotient, 3, and the remainder, 389456. Now, in order to find the *tenths* figure, we remark, that instead of multiplying the remainder by 10, (*i. e.*) placing a 0 to the right of it, we can divide the divisor by 10; that is, suppress one 0 on its right. Performing then the division, we have 3 for quotient, expressing *tenths*, and the remainder, 81356. In the same manner, instead of putting a 0 to the right of this remainder, we suppress a second 0 on the right of the divisor, and divide 81356 by 10270; applying still, if we wish, the rule of (38). We obtain thus the new quotient, 7, and the remainder, 9466.

Suppressing the last 0 on the right of the divisor, we divide 9466 by 1027; this gives the quotient, 9, and the remainder, 223. Setting out from this remainder, we follow the rule in (92), in order to obtain the remaining decimal figures. This second method is obviously less simple than the first; and we mention it, because it gives us the opportunity of showing how to operate when we have zeros to annex to the remainders of a division, of which the divisor is terminated by one or more zeros.

95. *Particular Cases.* — When there are no decimal places in one of the terms of the division — For example, we can have 51·47876 to be divided by 849, or 3145 to be divided by 23·479.

In the first of these examples, we would proceed according to the first method indicated in (94), under the head *secondly*.

In the second, we suppress the point in the divisor, and annex to the dividend as many zeros as there are decimals in the divisor. This is the same thing as multiplying both terms by the same number. These cases are too simple to demand farther development.

CONVERSION OF VULGAR FRACTIONS INTO DECIMALS.

96. We have seen in (92) how we are led to convert a vulgar fraction into a decimal. This operation forms an essential part of the theory of the division of decimal fractions. But we will make here an important observation, which shall serve us in the

exposition of the properties of decimal fractions, which we have to establish hereafter.

This observation consists in this, *that instead of placing zeros to the right of the different remainders which we obtain by applying the rule of* (92), *we can place at once these zeros on the right of the dividend, and perform the division of the resulting number by the divisor, taking care to place the point in the place to which it belongs in the quotient.*

In order to establish this second method of proceeding, we take the example, $\frac{13}{47}$, and write out both methods.

$$\begin{array}{r|l} 130 & 47 \\ \hline 360 & 0{\cdot}276595 \\ 310 & \\ 280 & \\ 450 & \\ 270 & \\ 35 & \end{array} \qquad\qquad \begin{array}{r|l} 13000000 & 47 \\ \hline 360 & 0{\cdot}276595 \\ 310 & \\ 280 & \\ 450 & \\ 270 & \\ 35 & \end{array}$$

In the first method, after writing a zero in the quotient, to take the place of the entire number, we annex a zero to the numerator, 13, of the fraction, in order to obtain the *tenths;* we then place another zero to the right of the remainder of this division, in order to obtain hundredths, and so on, until the total number of zeros thus successively brought down is six. In the second method, we multiply the numerator 13 by 1000·000 first, and then perform the division. It is obvious that the quotient thus obtained differs from that obtained by the first method of proceeding, in being 1000000 times greater, and that we reduce it to its true value by dividing it by 1000000, or by pointing off six decimal figures on the right.

DECIMAL SYSTEM OF WEIGHTS, MEASURES, AND COINS.

Having now discussed the four fundamental operations of arithmetic in their application to decimal fractions, we can appreciate the advantages which the calculus of decimal fractions presents over that of vulgar fractions, and are prepared to judge how important it is to establish a *decimal* system of weights, coins, and measures. In the United States we have the decimal system of coins in the Federal money. In France, the decimal system of weights, coins, and measures, has, after many efforts, been established, in spite of the obstacles occasioned by ignorance and prejudice. We give these decimal systems, with a few examples, in order to illustrate their advantages over the ordinary systems, with their irregular subdivisions.

97. The denominations of the currency of the United States are *Eagles*, *Dollars*, *Dimes*, *Cents*, and *Mills*, (the last three terms expressing their relative values to the dollar by their derivation).

TABLE.

The Eagle is divided into	10 dollars.
The Dollar	10 dimes.
The Dime	10 cents.
The Cent	10 mills.

The dollar sign being ($), we would, for example, write 56 dollars, 57 cents, and 5 mills, simply $56·575. In order to make the comparison, if we wished to write £15 10*s*. 6*d*. in parts of a pound, we would have to write £15 $+ \frac{10}{20} + \frac{6}{12}$ of $\frac{1}{20}$. And in order to express this decimally, we would have to reduce the compound fraction to a simple one, and then the vulgar fractions to decimals by last article.

French Coins.

The *franc* is the principal unit of the new French system of coins, its divisions being the *decime* and *centime*. The Napoleon contains 20 francs. The *sou*, or piece of 5 centimes, is still retained, but all calculations are made with the franc and its decimal divisions.

TABLE.

The Franc is divided into	10 decimes.
The Decime	10 centimes.

Thus, we would write 545 francs, 8 decimes, (16 sous), and 4 centimes, 545·84 *fr.*

We will now explain the nomenclature of the French system of weights and measures, to which the name metrical system has been given, the *metre* being the principal unit.

98. The unit of length, to which we give the name *metre*, is the ten millionth part of the distance from the pole to the equator, measured on the meridian of Paris. According to measurements made and verified with the utmost precision, the *metre*, valued in old French feet and inches, is equal to 3 *feet*, 0 *inches*, 11·296 *line*, to within less than $\frac{1}{1000}$ of a line, or equal to 39·3809171 of our inches.* In order to designate measures smaller or larger than the metre, it is agreed upon to employ the following prefixes, (taken from the Greek and Latin).

Myria, *Kilo*, *Hecto*, *Deca*, *Deci*, *Centi*, *Milli*, which signify *ten thousand*, *thousand*, *hundred*, *ten*, *tenth of*, *hundredth of*, *thousandth of*, (the multiples being indicated by the Greek prefix, the submultiples by the Latin). These prefixes are placed before the word *metre;* and the following table is formed. For convenience of comparison, we convert the divisions and subdivisions into parts of our inch.

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* This measurement of the arc of the meridian was made under the auspices of Arago and Biot. Several degrees, measured with great accuracy, served as a basis for the calculation of the length of the whole meridian.

Myriametre, or 10,000 metres = 393809·171 *inches.*
Milometre, " 1000 metres = 39380·9171 "
Hectometre, " 100 metres = 3938·09171 "
Decametre, " 10 metres = 393·809171 "
Metre, principal unit = 39·3809171 "
Decimetre, = $\frac{1}{10}$ of a metre = 3·93809171 "
Centimetre = $\frac{1}{100}$ of a metre = 0·393809171 "
Millimetre = $\frac{1}{1000}$ of a metre = 0·0393809171 "

N. B. The *myriametre*, and the *kilometre*, are the itinerary measures at present adopted in France. The myriametre is 6·22 miles.

MEASURES OF SUPERFICIES; OR, SQUARE MEASURE.

99. The natural unit of surface is the *square metre;* that is, a square which has a metre for its side. The *decimetre squared*, or the square which has a *decimetre* for its side, is $\frac{1}{100}$ of the metre squared; the square centimetre is $\frac{1}{10000}$, and so on, for the rest. The *square decametre* is equal to 100 square metres. This measure we take for the *principal unit* in all field measures; and this unit is called *are.* The multiples and subdivisions of the *are* are also designated by the aid of the prefixes, *myria, hecto, deci, centi* . . . . Thus,

The Myriare .... = 10,000 ares =
Kilare .... = 1000 ares
Hectare .... = 100 "
Decare .... = 10 "
Are = the principal unit = 100 *sq. ms.* = 119·665 *sq. yds.* = $\frac{1}{4}$ acre, about.
Deciare .... = $\frac{1}{10}$ of an are.
Centiare .... = $\frac{1}{100}$ of an are.
Milliare .... = $\frac{1}{1000}$ of an are.

N. B. The *myriare,* the *hectare, are,* and *centiare,* are the only measures used. The *centiare* is the *square metre.**

MEASURES OF VOLUME.

100. The unit of volume is the *cubic metre;* that is, a cube, (solid, of the form of a die), which has a *metre* for its *side.* The multiples and submultiples of the cubic metre have as yet received no particular names. The 1000th of the cubic metre is called the *cubic decimetre,* because it is a cube with a decimetre for its side, &c., for the cubic centimetre . . . . When the measures of volume are applied to wood for burning, or to materials of building, the principal unit or cubic metre is called *stere.* We then have the *decastere,* or measure of ten *steres.* The stere = 35·375 *cubic feet.*

MEASURES OF CAPACITY, BOTH DRY AND LIQUID.

101. The unit of capacity is the *cubic decimetre,* which is called *litre.*

As to the decimal multiples and submultiples, we give those which are chiefly used.

Hectolitre .... = 100 litres.
Decalitre ..... = 10 litres.
Litre .... = principal unit = 1·057 U. S. qts. = 61.074 [cub. in.
Decilitre ...... = $\frac{1}{10}$ of a litre.
Centilitre ..... = $\frac{1}{100}$ of a litre.

WEIGHTS.

102. The unit of weight is the weight of a *cubic centimetre* of distilled water, at the temperature of maximum density, viz., 39·5° Fahrenheit. The name given to this unit is *gramme.* The gramme is equal to 0·002204737 pounds avoirdupois.

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* A partial decimal square measure has been introduced among surveyors in the United States. The surveyor's chain, 66 feet in length, is divided into 100 equal links; and we have

10,000 square links = 1 sq. chain.
10 square chains = 1 acre.

TABLE.

		lbs.
The Myriagramme is ……	=10,000 grammes=	22·04737
Kilogramme ……….	= 1000 grammes=	2·204737
Hectogramme ………	= 100 grammes=	0·2204737
Decagramme ………	= 10 grammes=	0·02204737
Gramme ……………	= principal unit=	1·002204737
Decigramme ……….	= $\frac{1}{10}$ of a gramme=	0·0002204737
Centigramme ………	= $\frac{1}{100}$ of a gramme=	0·00002204737
Milligramme ………	= $\frac{1}{1000}$ of a gra. =	0·000002204737

N. B. The half kilogramme is about equal to the old French pound, nearly equal to our pound avoirdupois.

103. Such is the nomenclature of the measures which compose the *metrical system*. We can now judge of the advantages which this system possesses over the ordinary measures.

1st. It is uniform and simple, inasmuch as its principal units and their subdivisions follow the law of the decimal system of numeration.

2d. It is fixed, invariable, and susceptible of being adopted in all countries, since it is equally adapted to any climate or latitude.

All these measures have for their base one primitive measure, the metre, which is taken from the dimensions of the earth itself.

We will dwell but little upon the application of the four fundamental operations of arithmetic to the decimal system of weights and measures, since every collection of principal units and their subdivisions, according to the nomenclature, can be expressed by a decimal fraction; and, therefore, these operations become operations upon decimal fractions, considered as abstract numbers. For these last operations we have already established fixed rules. Nevertheless, we will propose some questions in multiplication and division, because they will afford opportunity for some important remarks upon approximate calculations.

Examples under the different tables illustrating the above.

1st. — 56 kilometres, 25 decametres, 5 metres, and 9 millimetres, are written, 56255·009 metres.

2d. — 25 hectares, 4 ares, and 6 centiares, are written 2504·06 ares.

3d. — 34 hectolitres, and 6 centilitres, are written 340·06 litres.

4th. — 54 myriagrammes, 4 decagrammes, 7 decigrammes, and 3 milligrammes, are written 540040·703 grammes

MULTIPLICATION.

104. *Question first.* — *Required, the price of* 35 *metres,* 429 *millimetres of a certain stuff, one metre of which costs* \$19 *and* 76 *cents.*

Here, if we multiply 35·429 *m.* by \$19·76, we will obtain a product which, expressed in dollars, cents, and mills, will be the price required. The abstract product of these numbers (93), is 700·07704; then, \$700·07, or, more exactly, \$700·08 is the price of 35·429 *m.* Sometimes the fraction of the metre is expressed by a vulgar fraction. In this case, the operation can be performed in two ways.

Question second. — *What is the price of* $23\frac{3}{4}$ *m. of a piece of stuff, at* \$8·25 *cts. per metre?*

1st. The reduction of $\frac{3}{4}$ to decimals, gives 0·75; the question is then reduced to multiplying 8·25 by 23·75, which gives 195·9375; then, \$195·94 *cts.* is the price of the $23\frac{3}{4}$ metres, to within less than $\frac{1}{2}$ cent (93).

2d. We could also operate as follows:

$$\begin{array}{r}
8{\cdot}25 \\
23\frac{3}{4} \\
\hline
24{\cdot}75 \\
165{\cdot}0 \\
\hline
189{\cdot}75 \\
\frac{1}{2} = 4{\cdot}125 \\
\frac{1}{4} = 2{\cdot}0625 \\
\hline
195{\cdot}9375
\end{array}$$

In this operation, after forming the product of the two entire parts, we have added the two partial products, and placed the point where it properly belongs, in order to avoid all error in the final result. We have then multiplied 8·25 by $\frac{3}{4}$ ($\frac{1}{2}+\frac{1}{4}$) by taking first the half of 8·25, which gives 4·125; then the half of this half, which gives 2·0625. Now, taking the sum, we get 195·8375, as by the first method.

This last method of proceeding is preferable, when the *vulgar fraction* cannot be converted into a *limited* number of decimal figures.

Third Question.—To find the price of 89$\frac{11}{12}$ metres, supposing one metre to cost $47·19.

1st operation,

$$\begin{array}{r} 47\cdot19 \\ 89\frac{11}{12} \\ \hline 424\cdot71 \\ 3775\cdot2 \\ \hline 4199\cdot91 \\ \frac{6}{12}=23\cdot595 \\ \frac{3}{12}=11\cdot7975 \\ \frac{2}{12}=7\cdot8650 \\ \hline 4243\cdot1675 \end{array}$$

Then, 89$\frac{11}{12}$ metres cost $4243·17, to within less than one cent.

Otherwise, commencing by converting $\frac{11}{12}$ into decimals, we find 0·91666 . . . .; and we must multiply 89·916666 . . . . by 47·19.

89·91	89·916	89·9166
47·19	47·19	47·19
80919	809244	8092494
8991	89916	899166
62937	629412	6294162
35964	359664	3596664
4242·8529	4243·13604	4243·164354

This table gives three distinct operations. 1st, with two decimal figures of the multiplicand; 2d, with three; 3d, with four;

and we see it is the last only which gives the approximation to within less than one *cent.*

The difficulty here is to know how many of the decimal figures of the multiplicand we must take, in order to be assured that we have the degree of approximation required by the nature of the question; while by the first method we obtain a complete result, of which we can, according to choice, neglect more or less of the decimal figures.

N. B. We could also reduce $89\frac{11}{12}$ to a single fraction; then multiply 47·19 by this fraction; an operation longer than the first method which we have used.

DIVISION.

105. *Question Fourth.—A piece of land containing* 23 *hectares,* 9 *ares,* 25 *centiares,* (23 *h.*, 0925 *c.*), *was bought for* $83,719·25. *Required the value of the hectare?*

We must here divide 83719·25 by 23·0925; and the quotient, valued in dollars and cents, will represent the price per hectare.

We obtain, by simple division of decimals, $3625·38.

Question Fifth.—$28\frac{19}{24}$ *kilogrammes, of a certain material, cost* $519·35. *What is the price per kilogramme?*

Here we may use two methods. 1st, Reduce $28\frac{19}{24}$ to a single fractional number, giving $\frac{691}{24}$. Then multiply 519·35 by $\frac{691}{24}$, inverted, (Art. 59); we thus find for result, 18·038.

2d. We convert $\frac{19}{24}$ to decimals, which gives 0·79166 . . . .; then we divide 519·35 by 28·79, taking only two decimal places of the divisor; we obtain thus, 18·039. Then, $18·04 is the price per kilogramme of the stuff.

These examples suffice to show how we must proceed in the multiplication and division of denominate numbers of the decimal systems, and to show how much simpler these operations are than in the ordinary systems of compound numbers.

We will add here, as belonging properly to the preceding theories, some notions upon the different divisions of the circle and thermometer.

106. *Of the two divisions of the Circle.* — The circumference of a circle is defined in geometry a recutiant line, all the points of which are equally distant from a point within, called the centre. In all the scientific works in this country, the circumference is divided into 360 equal parts, called *degrees* (°); each degree into 60 equal parts, called *minutes* (′); each minute into 60 equal parts, called *seconds* (″). This is called the sexagesimal division. When the French reformed their system of weights and measures, they adopted also a *centesimal* division of the circumference of the circle, the use of which is becoming very general among the scientific men of Europe. In this new *centesimal* system, the circumference is divided into 400 equal parts, called degrees (°); each degree into 100 equal parts, called minutes (′); each minute into 100 parts, called seconds (″); each second into 100 equal parts, called thirds (‴), &c.

Example of Sexagesimal Division. — 45 degrees, 38 minutes, 25 seconds, are written 45° 38′ 25″.

Example of Centesimal Division. — 28 degrees, 56 minutes, and 23 seconds, are written 28·5623°, in the decimal form. In order to reduce the divisions of the sexagesimal system to a compound number of the centesimal, we observe that the quarter of the circumference, called a quadrant, is in one system 90°, and the other 100°. Then, 1° sexagesimal $= \frac{100}{90}$, or $\frac{10}{9}$ of a degree centesimal, and *vice versâ;* 1° centesimal $= \frac{9}{10}$ of a degree sexagesimal.

We are thus led to the two following rules:

1st. To convert a compound number sexagesimal to a compound centesimal. *Reduce, first, to a fractional number of degrees* (70); *then multiply this number by* $\frac{10}{9}$, *and convert the result into decimals.* The entire part will express the centesimal degrees; the decimal part, divided into periods of two figures each, the minutes, seconds, &c.

2d. Reciprocally, to convert a compound centesimal number into a compound sexagesimal. *Subtract from the given number, expressed in decimal form,* $\frac{1}{10}$ *of this number, (or simply take* $\frac{9}{10}$ *of it).* The entire part of the result will represent the number of sexagesimal degrees. The decimal part we convert into minutes and seconds by the known rules for converting fractions of a higher denomination into units of a lower.

Examples. — 1st. Convert 34° 59′ 17″ sexagesimal, into degrees, minutes, and seconds, centesimal. — 34° 59′ 17″, converted to seconds, give 125957″, or $\frac{125957}{3600}$ of a degree; this, multiplied by $\frac{10}{9}$, gives $\frac{125957}{3240}$. Finally, the division of 125957 by 3240, gives 38·875617 . . . . or 38° 87′ 56″ 17‴ centesimal.

Reciprocally, 2d. — To convert 38°·875617 centesimal, into degrees, minutes, and seconds, sexagesimal.

$$\begin{array}{rr} & 38{\cdot}8756170 \\ \frac{1}{10} & 3{\cdot}8875617 \\ \hline & 34{\cdot}9880553 \\ & 60 \\ \hline & 59{\cdot}283318 \\ & 60 \\ \hline & 16{\cdot}99908 \end{array} \quad \text{or, } 34^\circ\ 59'\ 17''.$$

OF THE PRINCIPAL DIVISIONS OF THE THERMOMETER.

107. The thermometers mostly used on the continent of Europe are, the thermometer of Réaumur, and the Centigrade. In England and the United States, the use of Fahrenheit's thermometer is almost universal. These all differ in their scales of subdivision only. In Réaumur's, the interval between the freezing and boiling points of water is divided into 80 equal parts, called degrees of Réaumur; in the Centigrade, this same interval is divided into 100 parts, called *centesimal degrees.* It follows, that each degree of Réaumur's is equal to $\frac{100}{80}$, or $\frac{5}{4}$, of the Centigrade degree; and, reciprocally, each Centigrade degree is equal to $\frac{4}{5}$ of the degree of Réaumur. Moreover, the fractions of the degree are expressed generally in both by decimal fractions. Thus, it is a very simple matter to transform one into the other.

1st. In order to convert a decimal number of degrees of Réaumur into Centigrade degrees, we add to the number one-fourth of itself. The result of the addition is the number sought.

2d. In order to convert a decimal number of centesimal degrees into degrees of Réaumur, subtract from the given number one-fifth of itself, and you have the number sought.

Thus, for example:

$$39°\cdot4716 \text{ R.} = 39\cdot4716 + 9\cdot8679 = 49°\cdot3395 \text{ C.}$$

Reciprocally,

$$49°\cdot3395 \text{ C.} = 49\cdot3395 — 98679 = 39°\cdot4716 \text{ C.}$$

In Fahrenheit's thermometer, the freezing point of water is 32°, instead of 0°, and the interval between that and the boiling point (212°) is 180°. Then, the degree of Fahrenheit is $\frac{100}{180} = \frac{10}{18}$, or $\frac{5}{9}$ of the degree Centigrade; and, reciprocally, the degree Centigrade is $\frac{9}{5}$, or $\frac{18}{10}$, of the degree of Fahrenheit. In the actual reduction from one of these scales to the other, we must always keep account of the different start point, both for negative and positive temperatures. Thus,

1st. To convert a decimal number of degrees Fahrenheit (+) into centesimal degrees, we must first subtract 32°; then remove the decimal point one place farther to the right, and divide by 18, (or multiply by 5 and divide by 9).

2d. To convert a decimal number of degrees Centigrade into degrees of Fahrenheit, remove the decimal point one place to the left, and multiply by 18, (or multiply by 9, and divide by 5); then add 32° to the result.

Example 1*st.* To convert 56°·259 Fahrenheit into Centigrade degrees.

$$56°\cdot259 — 32° = 24°\cdot259 \ldots\ldots 24°\cdot259 \times \frac{10}{18} = \frac{242\cdot59}{18} = 13°\cdot477 \text{ C.}$$

2*d.* To convert 13°·48 C. to degrees Fahrenheit.

$$13\cdot48 \times \frac{18}{10} = 1\cdot348 \times 18 = 24\cdot259 \ldots\ldots 24\cdot259 + 32° = 56°\cdot259 \text{ F.}$$

The rules for the conversion of the —— (minus) degrees, and also for conversion of Fahrenheit into Réaumur, are too obvious to discuss them farther.

108. General Conclusion. — This first part of our work includes all which constitutes elementary arithmetic, the principal object of which is the exposition and development of the methods to be followed, in order to perform upon numbers all possible operations. These operations are to the number of four fundamental ones, *addition*, *subtraction*, *multiplication*, and *division*. All the others, such as the reduction of fractions to the same denominator, to their simplest form, the conversion of a vulgar fraction into a decimal, &c., are nothing more than combinations of those which we have just given.

There are two other species of operation, or rather two particular cases of the last two fundamental operations, of which we have not spoken; because, in order to be developed in a complete manner, they require some knowledge of algebra. These are the *formation of powers*, and the *extraction of roots* of numbers. The *powers* of a number are the products which arise from the continued multiplication of a number by itself. Thus, $4 \times 4 \times 4 \times 4 \times 4 =$ the 5th power of 4. The formation of powers is evidently then a particular case of multiplication. The *roots* of a number are those numbers whose continued products, each by itself, will produce the given number. Then, the extraction of roots proposes the solution of the problem — *Given a number, to find the two equal factors which form it, or the three equal factors*, &c.; evidently a particular case of division. We will not discuss these, because they are fully treated in all of the good text-books on algebra.

In the next chapter, we propose to consider numbers in a general manner, independently of every system of numeration, and to develop the properties belonging to any given system. This will be in some sort *Arithmetic Generalized.*

SECOND PART.

CHAPTER V.

GENERAL PROPERTIES OF NUMBERS.

109. Introduction. — Before going farther into the science of numbers, and in order to investigate their properties with more facility, we must borrow from algebra some of its materials, such as letters and signs (some of which we have used already), by the aid of which we indicate, in a general and abridged manner, the operations and the reasoning which the resolution of a question requires.

1st. The letters, which we employ instead of figures, in order to represent numbers. Their use affords at once a mode of writing, more concise and more general than that of figures.

2d. The sign + plus (already used), to indicate the addition of two or more numbers.

3d. The sign minus — (already used), to indicate the subtraction of one number from another.

4th. The sign of multiplication is ×, or a point, which we place between the two numbers, read *multiplied by*. Thus, $a \times b$, or, $a.\ b$, mean a multiplied by b.

N. B. We have already used both these signs. Now, when the numbers, the multiplication of which we wish to indicate, are expressed by letters, then this multiplication will be indicated also by simply writing one of the letters after the other, with no sign between; thus, ab signifies a multiplied by b. But this method cannot be employed when the numbers are indicated by figures; for, if we wrote the product of 5 by 6, 56, this notation would be confounded with fifty-six. In the case of figures, then, ×, or some such sign between the numbers, is necessary. Another sign of multiplication is the parenthesis ( ).

5th. The sign of division, either a bar (—), as already used in vulgar fractions, or a bar with two points, thus (÷), or simply two points. Thus, $\frac{24}{6} = 24 \div 6 = 24 : 6 = 24$ divided by 6

6th. The *Coefficient* is the sign which we employ, when a number denoted by a letter is to be added to itself several times. Thus, instead of writing $a + a + a + a + a$, which represents the number a added to itself four times, we write $5a$. We say, then, the coefficient *is the number written on the left of another number, denoted by one or more letters, to show how many times this number is taken, or the number of times plus one it is added to itself.*

7th. The exponent is the sign which we employ, when a number denoted by a letter is multiplied several times by itself. Thus, instead of writing $a \times a \times a \times a \times a$, or $aaaaa$, we write simply a^5, which signifies that a is multiplied 4 times by itself.

The exponent *is then a number written to the right, and a little above another number, or letter expressing a number, showing the number of times plus one that this number or representative letter is multiplied by itself.*

8th. The sign which expresses that two numbers are equal, already used (=), read *is equal to*, or simply equals.

The axioms applied in the operations on equations we annex to this definition, viz: If equals be added to, subtracted from, multiplied, or divided by equals, the results will be equal.

These preliminaries being established, we will take up again some of the subjects of which we have treated in the first part, in order to investigate them more thoroughly. We will arrive thus at new properties, and at means of simplifying or modifying the methods in the different operations of arithmetic.

[In order to give some idea of the use of these different signs, and of the simplicity of the algebraic language, we will make a few applications.

Let us suppose, first, that we wish to express that a number, represented by a, is to be multiplied 3 times by itself; that the product thus resulting is to be multiplied 3 times successively by b; and, finally, the new product is to be multiplied twice by c; we will simply write $a^4b^3c^2$.

If we wish to express that it is necessary to add this last result 6 times to itself, or multiply it by 7, we write $7a^4b^3c^2$.

In the same manner, $6a^5b^2$ is the abridged expression of 6 times the product of the 5th power of a by the second power of b.

$3a—5b$ is the abridged expression of the difference between the triple of a and the quintuple of b.

$2a^2—3ab+4b^2$ is the abridged expression of the double of the square of a, diminished by the triple product of a and b, and augmented by four times the square of b.

Let us now see how we can effect, upon quantities expressed algebraically, the fundamental operations of arithmetic. We will limit ourselves to the most simple cases—those to which we will have to refer in the latter part of this treatise.

Addition.—In order to add two numbers, a and b, we write simply $a+b$. In the same manner, $a+b+c$ indicates the addition of the numbers a, b, c: that results from the notation we have established. In the same manner, $a—b$ and $c+d—f$, added together, form the single quantity, $a—b+c+d—f$. If we had to add $a—b$ and $b—c$, we would write $a—b+b—c$. But as, on the one hand, b is added, and on the other subtracted, it follows that these two operations balance each other, and the expression is reduced to $a—c$.

SUBTRACTION. — In order to subtract b from a, we write $a - b$. In the same manner, if we wish to subtract c from $a - b$, we write $a - b - c$.

Let it be required to subtract the expression $c - d$ from the expression $a - b$. We can first indicate the subtraction thus: $a - b - (c - d)$. But if we wish to reduce the result to a single expression, we must reason as follows:

If we had to subtract c alone from $a - b$, the result would be $a - b - c$. Now, as it is not c, but c diminished by d, which is to be subtracted, the result, $a - b - c$, is too small by the number of units in d; thus, the result will be brought back to its just value by adding d to $a - b - c$, or writing $a - b - c + d$.

That is to say, in order to subtract one algebraic expression from another, we must write the one to be subtracted with the *signs of all its terms changed, after the other;* thus forming one single expression.

We find by this rule, and analogous reasoning,

$$3a - (2b - 3c) = 3a - 2b + 3c.$$

$$5a - 4b - (6d - f + g) = 5a - 4b - 6d + f - g.$$

MULTIPLICATION. — Required to multiply a^4 by b^3.

We write $a^4 \times b^3$, or simply a^4b^3.

But if we have a^5 to be multiplied by a^3, we observe that the number a, being 5 times a factor in the multiplicand, and 3 times a factor in the multiplier, ought to be $5 + 3$, or 8 times a factor in the product. Thus, we have $a^5 \times a^3 = a^8$; that is to say, *when the same letter enters into both factors of the multiplication, we write it once in the product, and give it for exponent the sum of its exponents in the two factors.*

Required, now, to multiply $a - b$ by c.

We can first indicate the product in this manner $(a - b)c$. But, if we wish actually to perform the operation, we remark, that to multiply $a - b$ by c (27), is to multiply c by $a - b$; that is, to take c $a - b$ times, or as many times as there are units in a, diminished by the units in b. If, then, we multiply first c by a, which gives ac or ca, the product is too great by the product of c by b, or bc. Thus, we must subtract bc from ac, and we obtain $ac - bc$ for the required product, $(a - b)\,c = ac - bc$.

Required, again, to multiply $a - b$ by $c - d$.

The product can first be indicated thus: $(a - b)\,(c - d)$.

But, in order to obtain a single expression, we commence by multiplying $a - b$ by c, which gives $ac - bc$; and we observe, then, that it is not by c alone that we have to multiply $a - b$, but by c diminished by d.

Thus, the product $ac - bc$ is too large by the product of $a - b$ by a; that is, by $ad - bd$. Then, in order to reduce the product to its just value, we must subtract $ad - bd$ from $ac - bc$; which gives, by the rule of subtraction, $ac - bc - ad + bd$.

Examining this product, we deduce the following rule:

In order to effect the multiplication of two algebraic expressions, multiply successively each term of the multiplicand by each term of the multiplier; observing, that if two terms of the multiplicand and multiplier are affected with the same sign, their product is affected with the sign + (plus); but if they are affected with different signs, their product is affected with the sign — (minus).

DIVISION. — We will consider only a single case of this operation, in which the two terms of the division contain the same letters.

Required to divide a^7 by a^3.

We can first indicate the quotient in this manner: — $\frac{a}{a^3}$, or $a^7 \div a^3$. But a^7 is the product of which a^3 and the quotient are the two factors; hence, the exponent, 7, of the dividend, ought to be equal to the sum of the exponent of the factor known, 3, and of the unknown exponent of the quotient; then, reciprocally, this last *is equal to the difference between the exponent of the dividend and the exponent of the divisor;* that is, to $7 - 3$ or 4.

Thus, $$\frac{a^7}{a^3} = a^4. \quad \frac{a^3b^2}{a^2b} = a\,b \;.\;.\;.\;.\; \&c.$$

Such are the general notions of algebra, of which we will have to make use in the fifth and following chapters.]

Theory of Different Systems of Numeration.

110. We have seen (Art. 5), how, by the aid of ten characters or figures we can represent all numbers, setting out with the conventional principle, that every figure placed on the left of another, expresses units ten times greater than those of the first figure. We now propose to show that we can write all numbers with more or less than *ten* characters, provided we do not use less than two, (zero, 0, being always one of these characters).

We call, in general, the number of figures employed, the *base* of the system. The system in which two figures are used, viz: (10), is the *binary* system, and 2 is the base. The ternary system, of which 3 is the base, makes use of 3 figures, 1, 2, 0; the quaternary has four figures, 1, 2, 3, 0; the quinary, five, 1, 2, 3, 4, 0; &c., &c.

The base may be greater than ten; we must then have recourse to additional characters. Thus, in the system of which *twelve* is the base, the *duodenary* or *duodecimal*, we will have to use two new signs, α and β, to express *ten* and *eleven* numbers less than the base.

In every system *analogous* to the decimal system, the conventional principle holds that *every figure placed on the left of another, expresses units as many times greater than those of the first figure as there are units in the base of the system.* Thus, in the binary system, each figure acquires a value twofold greater for each place that it is removed to the left. In the ternary system, they increase in a threefold ratio; and, in general, in a system of which b is the base, a figure goes on increasing in a *b-fold proportion*, as it is removed one or more places to the left.

When a number is written in a system whose base is b, the first figure on the right expresses the units of the *first order;* the figure immediately on its left the units of the *second order;* the next figure on the left the units of the *third order;* and so on. It requires b units of the first order to make one of the second; b units of the second to form one of the third, &c.

111. We pass, now, to the manner of expressing in figures any entire number, whatever be the system which we adopt. In order to fix our ideas, we will consider the septenary system, which makes use of the seven characters, 1, 2, 3, 4, 5, 6, 0. Adding unity to *six,* we obtain *seven,* or the *unit of the second order;* which, according to the principle enunciated above, can be expressed by 10; since the 0, having no value of itself, makes the figure 1 at its left express one unit of the second order, or *seven* simple units. Placing, successively, all the figures of the system in the first and second place, we will evidently form all the consecutive numbers comprised between 10 = seven and the number expressed by 66.

For example, 11, 12, 13, 14, 15, 16, represent the numbers eight, nine, ten, eleven, twelve, thirteen, 20 = fourteen, 21 = fifteen, &c.

After reaching the number 66, if we add to it a new unit, there will result 6 units of the second order, plus seven units of the first order; that is to say, seven units of the second order, or a single one of the *third order,* which can be expressed by 100. Placing, successively, in the first, second, and third places, the different figures of the system, we will form all the consecutive numbers comprised between 100 and the number expressed by 666. Reasoning on this last number, as upon 66, we shall arrive at the unit of the *fourth order,* which is expressed by 1000; then we obtain, successively, all the consecutive numbers comprised between 1000 and the number expressed by 6666; and so on, to infinity; whence we see that all possible entire numbers can be written in this system. The same reasoning applies to any other system. Whatever system be adopted, the units of the different orders are respectively represented by 1, 10, 100, 1000, 10000, &c., as in the decimal system.

112. N. B. We have said (110), that the character 0 was indispensable in every system analogous to the decimal system; that is to say, in a system where the relative value of a figure depends upon the place which it occupies on the left of several

others. To speak rigorously, we could do without it; but the system would be less regular, as we shall see.

Let it be proposed, for example, to establish the ternary system, using the three significant figures, 1, 2, 3. The first three numbers are expressed by these figures. In order to represent *four, five,* and *six,* it would suffice to write 11, 12, 13. In order to express *seven, eight, nine, ten, eleven, twelve,* we would write

21 22 23 31 32 33

In the same manner,

111 112 113 121 122 123

would express

thirteen, fourteen, fifteen, sixteen, seventeen, eighteen.

It is not necessary to go farther, in order to see the inconveniences of this system. Its principal fault consists in this, that units of the same order are expressed in a different manner. Thus, in 13 and 23, the figure 3 expresses a unit of the second order, the same with the figures 1 and 2 on its left. In 123, 23 express *nine,* or units of the third order, the same with the figure 1 to the left of them. (The same process might be applied to the decimal system as is here applied to the ternary, by using a single character for *ten,* and dropping the 0.)

In making use of 0, it suffices to determine the number of units of different orders which enter into the proposed number, and to write, one after the other, the figures which express these units.

113. The perfect adaptation of the nomenclature of numbers, and the manner of writing them in figures, in the decimal system, permits us to write them easily from dictation in ordinary language. The same thing would be true of every system of numeration which had a special nomenclature appropriate to the system; in other words, a spoken numeration corresponding to its written one. But other systems do not present this immediate connexion with the nomenclature now in use.

Let it be proposed, for example, to express the number, *three hundred and sixty-nine*, referred to the decimal system in the septenary system. It is difficult to see, *à priori*, which are the figures proper to express units of the first, second, third . . . . order which it contains.

Now, since this number, written in figures in the decimal system, is three, six, nine, it follows that the question above depends on the following, which is much more general:—*A number being enunciated in ordinary language, or written in the decimal system, required to express this same number in the system whose base is b.*

In order to resolve it, we remark, that since it takes b units of the first order to make one unit of the second order, as many times as the proposed number contains the number b, so many units of the second order of the system, whose base is b, will it contain; that is to say, that if we divide this number by b, the quotient will express units of the second order, and the remainder, which will necessarily be less than b, will express the units of the first order of the number written in the system whose base is b.

In the same manner, since b units of the second order in the system whose base is b, form one unit of the third order in the same system, if we divide the quotient which expresses units of the second order by b, the new quotient which we thus obtain shall express units of the third order, and the remainder, always less than b, shall represent the units of the second order written in the system whose base is b, and so on for the rest.

Whence we see, that in order to pass from the decimal system to the system whose base is b, we must, 1st, *divide the given number by the base of the new system written in the decimal system, and write the remainder of this division apart, as expressing the units of the first order in the new system;* 2d, *divide the quotient obtained by the same base, and write the second remainder to the left of the first, as expressing the units of the second order;* 3d, *divide the second quotient by the same base, and write the third remainder on the left of the two preceding, because it ex-*

presses units of the third order; continue this series of operations until we arrive at a quotient smaller than the base of the new system; this last quotient expresses the units of the highest order, and is written on the left of all the remainders successively obtained.

Let us apply this rule to the number, 369, which we wish to express in the septenary system.

7 | 369
7 | 52 (5 1st rem.
7 | 7 (3 2d rem.
1 (0 3d rem.

Dividing 369 by 7, we obtain for quotient, 52, with remainder, 5, which we write apart, in order to express the units of the *first order* in the new system.

Dividing 52 by 7, we find 7 for quotient, and 3 for remainder, which we write to the left of 5, as it expresses units of the *second order.*

Dividing 7 by 7, we have 1 for quotient, and 0 for remainder, which indicates that there are no units of the *third order;* but we write a 0 to take the place.

Finally, as the quotient 1 is smaller than 7, it expresses the units of the fourth order, and the number in the septenary system is (1035).

On examining this operation, we shall find that we have obtained the *three hundred and forty threes*, the *forty nines*, the *sevens*, and the *units*, which the given number, 369, contains Hence, we might also proceed by the following rule:

Find, by inspection, the highest denomination of the new system which the given number contains; divide the given number by the number expressing the highest order of units written in the decimal system. Set the quotient apart, as expressing the highest order of units of the required number in the new system. Divide the remainder by the number expressing the value of the next lower order, and place the quotient on the right of the first *one to express the units of the next highest order. Divide the remainder by the next lower, placing the quotient on the right of the two preceding, &c., &c.*

Thus, in the same example, 369 to be converted into its equivalent number in the septenary system. We see that *three hundred and forty-three* is the highest order of unit of the septenary system which it contains.

We divide by 343; the remainder, 26, by 49; the remainder, by 7: the last remainder, 5, being necessarily less than seven, the base of the system expresses units. So we write the quotients from left to right, commencing with the first obtained, and write the last remainder on the right of the last quotient. We obtain, as before, (1035). The first method given is, however, the best, especially for large numbers.

343)369(1
343
49)26(0
7)26(3
5 last rem.

We find, by this method, the number 5347 of the decimal system, equal to (12343) of the system which has *eight* for its base.

8 | 5347
8 | 668 (3 1st rem.
8 | 83 (4 2d rem.
8 | 10 (3 3d rem.
1 (2 4th rem. (12343)

Remark. — It can happen that the base of the new system is greater than *ten*, the base of the decimal system. In this case, we proceed as follows: — Required, for example, to convert the number 8423 of the decimal system into its equivalent number in the duodenary system. The figures of this system are 1, 2, 3, 4, 5, 6, 7, 8, 9, α, β, 0. (The two Greek letters, α and β, being employed to designate *ten* and *eleven* in the new system.)

$$\begin{array}{r|l} 12 & \overline{8423} \\ 12 & \overline{701} \quad (\beta \text{ 1st rem.} \\ 12 & \overline{58} \quad (5 \text{ 2d rem.} \\ & \overline{4} \quad (\alpha \text{ 3d rem.} \end{array}$$

The base *twelve* being expressed ($4\alpha5\beta$) by 12 in the decimal system, we divide 8423 by 12, which gives 701, and remainder, $\beta = 11$, in the decimal system. We write this β apart, as expressing *units of the first order*. Likewise, in the third division, we obtain for a remainder *ten*, which, in the new system, is expressed by α; we then write α to the left of the two figures already found. We obtain thus ($4\alpha5\beta$) for the equivalent of the given number in the new system.

114. *Reciprocally, a number being written in a system whose base is b, required to enunciate it in the spoken numeration of the decimal system; that is, to convert it into its equivalent in that system.*

In general, let . . . . *hgfdca* be a number expressed in the system of which b is the base; a, c, d, f, &c., expressing units of the first, second, third . . . . order, (and not being an indicated product), as in (4°, Art. 109). It results from the fundamental principle established in (110), that the figure denoted by c, expresses units b times as great as the same figure standing alone would express; then, its relative value can be represented by $c \times b$, or simply by cb (109). In the same manner, the figure a expresses units b times as great as *those* of the figure c: hence, its relative value is equal to the product of d by $b \times b$ or b^2, and can be expressed db^2. We could show, in like manner, that fb^3, gb^4, hb^5 . . . . are the relative values of the other figures. Then, the given number is expressed by

$$a + cb + db^2 + fb^3 + gb^4 + hb^5 + \ldots.$$

Giving to the base b and to the figures a, c, d, f, particular values, we effect all the operations indicated in this expression, and we shall obtain the number corresponding to the particular data, converted into the decimal system.

Required, for example, to convert the number 4367, written in the system of *eight* figures, back into the decimal system.

This number can, according to the expression above, be placed under the form

$$7+6\times 8+3\times 8^2+4\times 8^3.$$

We have at once, then,

$$\begin{array}{lcr} 7 & \ldots\ldots\ldots & = 7 \\ 6\times 8 & \ldots\ldots\ldots & = 48 \\ 3\times 8^2 & \ldots\ldots\ldots & = 192 \\ 4\times 8^3 & \ldots\ldots\ldots & = 2048 \\ \hline & & 2295 \end{array}$$

Adding these numbers, we have 2295 for the value of (4367) in the decimal system. We can verify the accuracy of this operation by the rule of (113).

$$\begin{array}{rl} 8 \mid 2295 & \\ 8 \mid \overline{286} & (7 \text{ 1st rem.} \\ 8 \mid \overline{35} & (6 \text{ 2d rem.} \\ \overline{4} & (3 \text{ 3d rem.} \quad (4367). \end{array}$$

And, reciprocally, this last operation can be verified by the preceding one, which we will enumerate generally thus:

Form, first, the different powers of the base, b, written in the decimal system; multiply then all the figures of the number, written likewise in the decimal system, as a, c, d, f, g, h, respectively, by 1, b, b^2, b^3, b^4, b^5. *Adding the partial products, we shall have the number required.*

Given, for example, the number $(4\alpha 5\beta)$ in the duodenary system, to be converted into its equivalent in the decimal system. Since α and β, written in the decimal system, are 10 and 11 respectively, this number can be placed under the form

$$11+5\times 12+10\times 12^2+4\times 12^3.$$

$$\begin{array}{lr} 11 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; = & 11 \\ 5 \times 12 \quad . \; . \; . \; . \; . \; . \; . \; . \; = & 60 \\ 10 \times 12^2 \quad . \; . \; . \; . \; . \; . \; . \; = & 1440 \\ 4 \times 12^3 \quad . \; . \; . \; . \; . \; . \; . \; = & 6912 \\ \hline & 8423 \end{array}$$

Then, (4α5β) equals 8423, written in the decimal system.

115. The two preceding rules lead to a third, more general, which has for its object to convert any number from a system whose base is *b*, into its equivalent in a system whose base is *c*.

Convert the number from the system b to the decimal system, by (114); *then from the decimal system to the system c, by* (113).

Required, for example, to convert the number (23104) of the system whose base is 5, to its equivalent in the duodenary system. We obtain, first, for this number, transformed into the decimal system, 1654; then, for this last, transformed into the duodenary system, (β5α). We can verify this operation by making the transformations in an inverse order.

N. B. The above transformation from the quinary to the duodenary system, could be effected directly, without the intervention of the decimal system, by performing all the operations required in the quinary system; the only difficulty of this mode of operating being the want of agreement between the written numeration of this system and the *spoken numeration*, so universally in use.

124. The methods of performing the four fundamental operations of arithmetic, upon numbers written in any system whatever, do not differ from those which have been established for the decimal system. We must only recollect the law which exists between the units of different orders, in order to be able to convert the units of any order into units of the order next higher or next lower.

In order to familiarize beginners with the different systems of numeration, we will propose an example of each of the four operations in the duodenary system.

1st. Required to add 3704α, β2956, 27βα5, 48αβ.

We find for the sum of the simple units *thirty-two;* that is to say, 2 twelves and 8 units; we then write 8 in the units column, and carry 2 to the column of units of the second order. The sum of the units contained in this second column is thirty-one, or 2 units of the third order, and 7 of the second; we write the 7, and carry the 2 to the next column. Operating in the same manner on the other columns, we obtain (15α678) for result.

3704α
β2956
27βα5
48αβ
15α678

2d. Required to subtract from 5α0046 (with ββ set above the two zeros)
The number, 47α68β
121577

As we cannot subtract β from 6, we borrow one unit of the second order from the 4, and say, β from *eighteen* leave 7. Passing to the next subtraction, as we cannot subtract 8 from 3, we borrow one unit from the first significant figure to the left. As there are two zeros between, we say, this unit thus borrowed equals twelve, or β + one of the next lower, which *one* equals β + one of the next lower, which *one* equals twelve of the same order with the 3. We then subtract 8 from fifteen, giving 7.

In the two following subtractions, we regard the zeros as replaced by β, and continue the operation to the end.

3d. Required to multiply 3407α
by 5α68
228528
180360
294664
148332
177608828

We premise here a table of multiplication as far as the figure β, the highest figure of the system, after the manner of the table of Pythagoras.

This being premised, we multiply 3407α by 8, and say; 8 times α make eighty, or (68) of duodenary system; we write the 8 and carry the 6. Then 8 times 7 make fifty-six, and 6 make sixty-two, or (52) of the duodenary. We write the 2 and reserve 5 for the next column. Continuing this operation, we obtain for a partial product, 228528. As to the products of the multiplicand by the other figures of the multiplier, the same reasoning applies, and we use the same processes as in the decimal system. Summing up the products, we obtain 177608828.

4th. Let us verify this operation by division. We simply divide the product obtained, by one of the factors. In order to obtain the number of units of the highest order in the quotient, we take the first five figures on the left of the dividend, and divide 17760 by 5α68. For, thus, we see that 17 contains 5 three times, with a remainder. Multiplying the divisor by 3, and subtracting the product from the first partial dividend, we obtain for a remainder, 1βα0. We bring down 8 and divide 1βα08 by 5α68, obtaining 4 for quotient, and 3α0 for remainder.

$$\begin{array}{l|l} 177608828 & 5\alpha68 \\ \hline 1\beta\alpha08 & 3407\alpha \\ \hline 3\alpha082 & \\ \hline 4\alpha968 & \\ \hline 0000 & \end{array}$$

When the following figure 8 is brought down, the new dividend does not contain the divisor; we then place 0 in the quotient and bring down 2, which gives 3α082 for the next partial dividend. Proceeding in the same manner with the rest, we obtain 3407α for the required quotient.

We can now see how we can pass at once from the number (23104) of the quinary system, to its equivalent in the duodenary (115). We must divide 23104 by 22, or *twelve* written in the quinary system, and perform the division in that system, we would thus obtain a remainder which would express the units of the first order in the duodenary system, and a quotient which we would divide again by 22, or *twelve* expressed in the *quinary* system, in order to get the units of the second order, &c., &c.

117. *General Remark.* — The duodecimal system offers some

advantages over the decimal, inasmuch as its base *twelve* contains a greater number of factors than *ten*. For *twelve* is divisible by 2, 3, 4, 6; while the only factors of 10 are 2 and 5.

Nevertheless, we could not substitute the duodenary system, or any other, for the decimal, without replacing the ancient nomenclature by a new one, which was more appropriate to the system adopted, that is, which made the enunciation of written numbers easier.

We shall perceive, moreover, that the greater part of the properties of numbers which have been discovered are true, whatever be the system of numeration which we adopt, and some, which shall seem to belong to the decimal system in particular, have their analogous properties in the other systems. The employment of the letters of the alphabet in order to represent numbers, is well calculated to make the generality of these properties appear, as they can express numbers enunciated in any system of numeration whatever.

PRINCIPLES OF MULTIPLICATION AND DIVISION. DIVISIBILITY OF NUMBERS.

118. WE have already demonstrated (25) and (26). 1st. That *to multiply a number by the product of several factors, is the same thing as multiplying the number successively by each one of the factors.*

2d. That *the product of two numbers is the same in whatever order we effect their multiplication.*

Though the reasoning were developed upon particular numbers, they are not the less rigorous on that account; and in order to convince ourselves, it suffices to go through it again, denoting the numbers by the letters a, b, c, &c.

We propose to verify only the accuracy of the second of the above propositions, *whatever be the number of factors to be multiplied together.* We commence by remarking, that if we had to multiply a number N by b, and then to multiply the product obtained by c, it will amount to the same thing to multiply N first by c, and then the product by b. *In other terms* (3), *in a multiplication of more than two factors, we can invert the order of the last two multiplications without changing their product*, or, $N \times b \times c = N \times c \times b$.*

For it results from the first principle above, that $N \times b \times c = N \times bc$; but in virtue of the second principle, we have $bc = cb$; then, $N \times b \times c = N \times cb$, or, $N \times b \times c = N \times c \times b$. Q. E. D.

From this proposition, and the proposition that the product of two numbers is the same in whatever order we take them, it is easy to deduce the same proposition for *three* numbers.

Let a, b, c, be the numbers proposed.

We say that $abc = bac = bca = cba = cab = acb$.

For the second product is equal to the first, in virtue of proposition 2d; the third is equal to the second, in virtue of proposition 3d; the fourth is equal to the third, in virtue of 2d; the fifth is equal to the fourth, in virtue of the 3d; finally, the sixth is equal to the fifth, in virtue of 2d. Then all the products are equal. From this demonstration for three factors, and from the incidental proposition (3), we deduce with the same faculty the proposition for four factors.

Let a, b, c, d, be the numbers proposed.

We say then, that

$$\begin{aligned} & abcd = bacd = bcad = cbad = acbd = cabd \\ &= abdc = \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \\ &= bcda = \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \\ &= cadb = \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \end{aligned}$$

Firstly, the six products of the first horizontal line are equal to each other, in virtue of the proposition for *three* factors, since they result from the multiplication of abc, bac, &c., &c., by the same number, d. The first product of the second line is equal to the first of the first line by reason of (3); as to the other products of this line, we dispense with writing them; they can be

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* We here for convenience sake, shall give different significations to $N \times b \times c$ and $N \times bc$, regarding the last as the *product performed* of b and c.

found easily, keeping c in the last place in each; they are all equal to the first by the proposition for three factors. We could thus proceed with the other two lines, applying alternately the incidental proposition (3), and the proposition for three factors. We thus prove the proposition for all possible products of a, b, c, d, since we cannot form more than 6 products terminated by the same letter. The same mode of demonstration can obviously be easily extended to any number of factors.

119. The demonstration which we have given of the preceding principle, supposes that the numbers upon which we are reasoning are entire numbers (Arts. 25 and 26); but if we reflect a little upon the rules established for the multiplication of fractions, we perceive that the property is equally applicable to fractional numbers. Moreover, this proposition completes the demonstration of the method established for the reduction of fractions to a common denominator given in the chapter on fractions.

Divisibility of Numbers.

120. The property which certain numbers possess of being exactly divisible by others, and the investigation of the divisors of a number, form one of the most important theories of arithmetic. This theory depends upon a series of principles, which we proceed now to develop successively.

We will first repeat some preliminary definitions which we have already given. We say that every entire number, which divides exactly another entire number, is called a *factor*, *divisor*, or *submultiple* of this number, and this last is called a *multiple* of the first. Every entire number which has no other divisor except itself and unity, is called an *absolute prime number*, or simply a prime number. Two entire numbers are *prime with each other* when they have no other common divisor besides unity, which is a divisor of every number.

It follows from this, that a *prime* number which does not exactly divide another number, *is prime with the latter*, as they can have no common divisor greater than unity.

121. *First Principle.* — *Every number*, P, *which divides exactly one of the factors of the product* A × B, *divides necessarily the product;* or, what amounts to the same thing, *every entire number which divides another exactly, divides necessarily the multiples of this number.*

For, let Q be the quotient supposed exact of the division of A by P; we have then $A = P \times Q$, whence, multiplying both sides by B, $A \times B = P \times Q \times B = P \times QB$; we see then that P is a factor of the product AB.

122. *Second Principle.—Every number which divides exactly the product of two factors, and which is prime with one of them, divides necessarily the other factor.*

Let A × B be the given product, P the number which divides this product exactly; we say that if P is prime with A, it will divide B.

For A and B being by hypothesis prime with each other, if we apply to them the rule of the greatest common divisor, we will be led to a remainder equal to 1; that is to say, denoting by $r, r', r'', \dots . 1$, the successive remainders, we will have the series of numbers.

$APr, r', r'', \dots 1$, A being greater than P; or, $PAr, r', r'', \dots 1$, if P is greater than A, for the different terms of the divisions to be performed. But, suppose that, before performing the operations, we commence by multiplying A and P by B, there will result the new series, $A \times B$, $P \times B$, $r \times B$, $r' \times B$ $\dots . 1 \times B$. Now, all these terms are divisible by P, since P is the common divisor of the two first terms. Then $1 \times B$, or B is divisible by P. Q. E. D.

N. B. It is important to remark that the proposition is only true when P is prime with one of the factors of the product.

For, if we have, for example, on the one hand 28 × 15, and on the other 12, which is not prime with either of the two factors of the product, the quotient of the division of (28 × 15), or 420 by 12, is exact and equal to 35, though 12 divides neither 28 nor 15. It is obvious in this case, that the two factors contain together all the prime factors which compose the divisor.

Thus we have

$$28\times 15=4\times 7\times 3\times 5=(4\times 3)\times 7\times 5=12\times 7\times 5=12\times 35$$

Consequence of Second Principle.—Any number whatever, P, *prime with all the factors except one of a product,* A × B × C . . . ., *can only divide the product, when it divides exactly the remaining factor.*

This is too obvious for discussion.

123. *Third Principle.* — *Every prime number,* P, *which divides exactly the product of two factors, divides one of them necessarily.*

For, suppose that P does not divide A, it is necessarily prime with A (120); then it must divide B (122).

From this result the following consequences.

124. 1st. *If a prime number,* P, *divides the product* A × B × C × . . . . . *of any number of factors, it divides one of the factors at least.*

2d. *Every prime number which divides the powers,* A^2, A^3, A^4, &c., *of any number,* A, *divides* A *itself.* For A^2, A^3, &c., being equal to A × A, A × A × A . . . . . P, can only divide these different products when it divides one of the factors.

3d. *If two numbers,* A and B, *are prime with each other, their powers,* A^2 and B^2, A^3 and B^3, &c., *are also prime with each other.* For any number, a, which is the common divisor of A^3 and B^3, for example, must divide A and B, which is, by hypothesis, impossible.

125. 4th. *Every number,* P, *prime with each one of the factors of a product,* A × B × C × . . . ., *is also prime with the product.* For suppose that a prime number, d, differing from 1, can divide at once P and the product A × B × C . . . . ., as d ought to divide one of the factors of the product, P would not be prime with this factor, which is contrary to the hypothesis.

126. 5th. *When a number,* N, *has been formed by the multiplication of several others,* A, B, C, D, ..., *this number can have no other prime factors except those which already enter into* A, B, C, D, &c. For every prime number which divides the product, A × B × C × D, and does not divide D, must divide A × B × C (123); in the same manner, every prime number which divides A × B × C, and does not divide C, must divide A × B, and, consequently, A or B. Thus, we can say in other terms, *a number being formed by the multiplication of several others, we cannot obtain it anew by multiplying numbers which contain prime factors different from those which enter into the numbers already multiplied.*

127. *Fourth Principle.—Every number,* N, *divisible by two or more numbers,* d, d′, d″, ..., *prime with each other, is divisible by their product.*

For, since d divides N, we have $N = d \times q$, q being an entire number; but by hypothesis, d' also divides N, then it divides $d \times q$, and, since d and d' are prime with each other, d' must divide q exactly; and we have $q = d' \times q'$, where q' is an entire number. Hence, $N = d \times d' \times q'$, and N is divisible by $d \times d'$. In the same manner, we can continue and show that N is divisible by $d \times d' \times d''$, and so on for the rest.

128. *Consequence.*—If d, d', d'', ..., numbers prime with each other, enter as factors into N a certain number of times, each denoted by n, n', n'', . . ., the number N is exactly divisible by $d^n \times d'^{n'} \times d''^{n''}$, and by all the numbers which we can obtain by multiplying two and two, three and three; the different powers of d, d', d'', ..., comprised between the first and the n^{th}, n'^{th}, &c , respectively. For, d, d', d'', ..., being prime with each other, d^n, $d'^{n'}$, $d''^{n''}$, ... must be so also; then (127), their products, two and two, three and three, must be exact divisors of N.

This principle serves as a basis in the investigation of all the divisors of a number.

It is useless to observe, that all the propositions established thus far are true in all the systems of numeration.

We will give some now which relate particularly to the decimal system.

129. *Signs of the divisibility of one number by others.*

There are certain signs or characteristics by which we can often tell whether a number is or is not divisible by others. A knowledge of these is often useful in practice.

The reasoning by which we will establish these signs or characteristics of divisibility, rests upon the following principle.

Let a number, A, be divided into two parts, B and C, so that we have

$$A = B + C \ (1).$$

1st. *If a fourth number,* D, *divide exactly the two parts,* B *and* C, *it divides also their sum.*

2d. *If the number,* D, *divides one of the parts,* B, *without dividing the other,* C, *it will not divide* A; *and the remainder of the division of* A *by* D, *is equal to that which the division of* C *by* D *gives.*

Of the first principle, it is very easy to give a general demonstration. Divide (1) by D, and we have

$$\frac{A}{D} = \frac{B}{D} + \frac{C}{D} \ (2).$$

Now the two terms, $\frac{B}{D}$ and $\frac{C}{D}$, are, by hypothesis, entire numbers; then the first number, $\frac{A}{D}$, must also be an entire number. As to the 2d principle, it is clear from the above equality (2), that if B is divisible by D, and C is not, A cannot be, for we would otherwise have a fraction equal to a whole number.

Again, B being divisible by D, we have $B = DQ$ (Q an entire number); C not being divisible by D, we have $C = DQ' + R$.

Then, $B + C = A = DQ + DQ' + R$, or $A = D(Q+Q') + R$. Whence we see that A divided by D, gives for quotient, $Q + Q'$, and remainder, R, of the division of C by D. These principles have been obviously assumed under the head of division.

130. Properties of the numbers 2, 5, 4, 25, 8, 125, ...

1*st.* *Every number terminated by one of the figures*, 0, 2, 4, 6, 8, *is divisible by* 2. For this number can be decomposed into two parts, viz: the part to the left of the simple units, and the collection of simple units. (For example, 38576 is equal to 38570+6). Now the first part being terminated by 0, is a multiple of 10; and 10 we know is divisible by 2; then this first part is divisible by 2. And if the second part contains 0, 2, 4, 6, 8, units exactly, it is divisible by 2. Hence, the number is divisible by 2.

If the number is terminated by one of the figures, 1, 3, 5, 7, 9, it is not divisible by 2, since one of its parts is divisible, and the other is not. The numbers divisible by 2, are even numbers, the others, odd numbers. The expression $2n$ (n being any entire number), embraces all the even numbers; the expression $2n + 1$, all the odd numbers.

2d. *Every number terminated by a* 0 (*zero*) *or* 5, *is divisible by* 5. The same demonstration as before, for 2. If the last figure is different from 0 or 5, the number is not divisible by 5; *and the remainder of the division of this number by* 5, *is equal to the remainder of the division of the last figure by* 5 (129). Thus, 1327 divided by 5, gives for remainder, 2, equal to the remainder of the division of 7 by 5. In the same manner, 34789 and 71436 give for remainders, 4 and 1.

3d. *Every number of which the two last figures, taken with their relative value, form a number divisible by* 4 *or* 25, *is itself divisible by* 4 *or* 25. For this number can be decomposed into two parts, the part on the left of the *tens*, and the part composed of tens and units. (For example, 3548 and 27875, are equal to 3500 + 48 and 27800 + 75). Now, the first part being terminated by two zeros, is a multiple of 100; 100 is divisible by 4 or 25, since 100 is equal to 25 × 4; then, this first part is also divisible by 4 or 25. Hence, if the second part is divisible by 4 or 25, then the whole number is. Thus, 3548 is divisible by 4, because 48 is a multiple of 4; 27875 is divisible by 25, because 75 is a multiple of 25. But 13758 is not divisible by 4, and gives for remainder, 2, equal to the remainder of the division of 58 by 4; 25659 is not divisible by 25, and gives for remainder, 9, or the remainder of the division of 59 by 25. For the number 25, the numbers terminating in 00, 25, 50, or 75, are the only numbers divisible by 25.

4th. *Every number, the three last figures of which considered with their relative values, form a number divisible by* 8 *or* 125, *is also divisible by* 8 *or* 125.

The demonstration of this is analogous to the preceding. It is founded upon the fact, that $1000 = 125 \times 8$.

131. *Properties of the numbers* 3 *and* 9. *Every number, the sum of whose figures is divisible by* 3 *or by* 9, *is itself divisible by* 3 *or by* 9. *And the remainder of the division of any number whatever by* 3 *or by* 9, *is the same as the remainder of the division of the sum of its figures by* 3 *or by* 9.

We remark, first, that a number which is composed of unity, followed by one or more zeros, is equal to a multiple of 9, increased by 1. (For example, $10 = 9 + 1, 100 = 99 + 1, 1000 = 999 + 1$, all the parts, 9, 99, 999, being divisible by 9 and by 3). It follows from this, that every number formed by a significant figure followed by one or more zeros, is itself a certain multiple of 9, augmented by this significant figure. For example,

$$70 = 7 \times 10 = 7(9 + 1) = 7 \times 9 + 7, 80000 = 8 \times (1000) = 8 \times (9999 + 1) = 8 \times 9999 + 8.$$

This established, let us take any number whatever, 6205473, for example. It can be decomposed in the following manner:

$$6000000 + 200000 + 0{\cdot}0000 + 5000 + 400 + 70 + 3.$$

And, according to what has just been said, it contains two principal parts. 1st. A sum of several multiples of 9, which sum is itself a multiple of 9 (129). 2d. The sum of the figures

$$6 + 2 + 0 + 5 + 4 + 7 + 3.$$

In other words, the number can obviously be written

$$6 \times 999999 + 6 + 2 \times 99999 + 2 + 0 + 5 \times 999 + 4 \times 99 + 4 + 7 \times 9 + 7 + 3.$$

The first part is divisible by 9. The second part is or is not. In the first case, the number proposed is divisible by 9; and in the second case, the remainder of the division of the significant figures by 9 is necessarily the remainder of the division of the whole number by 9. The demonstration for 3 is absolutely the same. We must make this observation, however, when the number is divisible by 9, it is necessarily so by 3; but it can be divisible by 3, without being so by 9.

To establish the proposition generally,

Let *gfdcba* be the given number, which we will denote, moreover, by N; we have, according to the fundamental principle of numeration, $N = a + 10b + 10^2c + 10^3d + 10^4f + 10^5g + \ldots$, an equation which can be placed under the form

$$N = \left\{ \begin{array}{llllll} & +(10-1)b & +(10^2-1)c & +(10^3-1)d & +(10^4-1)f & + \\ a & +b & +c & +d & +f & + \end{array} \right\}$$

by adding and subtracting b, c, d, &c., from the last number at the same time.

Now, according to what was premised above, $10-1$, 10^2-1, 10^3-1 . . . . being divisible by 3 or by 9, the first horizontal line is composed of a succession of numbers divisible by 3 or by 9. Thus, this first part of the number, N, is divisible by 9. Then, if the second part, which is nothing more *than the sum of the figures of the given number*, is divisible by 3 or by 9, the number itself is divisible by 3 or by 9; and, if this last part is not divisible by 3 or by 9, the remainder of this division will necessarily (129), be the remainder of the division of the number itself, by 3 or by 9.

N. B. In practice, instead of determining the sum total of the figures, in order to divide it by 9, we subtract 9 from the partial sum so soon as it exceeds or equals 9, as we proceed with the summing up, and continue the operation to the last figure. These partial subtractions do not obviously change the remainder, which we seek.

Example. — Given, the number

74683056743.

We say, 7 and 4 make 11; 9 from 11 leave 2; 2 and 6 make 8, and 8 make 16; 9 from 16 leave 7; 7 and 3 make 10; 9 from 10 leave 1; 1 and 0 and 5 make 6, and 6 make 12; 9 from 12 leave 3; 3 and 7 make 10; 9 from 10 leave 1; 1 and 4 and 3 make 8. Then, 8 is the remainder of the division of the number by 9.

132. *Property of the number* 11. — *Every number is divisible by* 11, *when the difference between the sum of the figures in the odd places, counting from the right, and the sum of the figures in the even places, is equal to* 0, *or divisible by* 11.

Before demonstrating this property, it is necessary to remark,

1st. *That every power of* 10 *of an even degree diminished by unity, gives a result divisible by eleven.*

For this result is necessarily composed of an even number of 9's, written one after another. Now, each division of two figures, taken separately, forms 99, or 9×11, divisible by 11; then, the numbers themselves are divisible by 11; or, in general, $10^{2n}-1$ is divisible by 11, ($2n$ expressing the even numbers).

2d. *Every uneven power of* 10, *augmented by unity, gives a result divisible by* 11.

For a power of an even degree of the number 10 can be expressed by 10^{2n+1} (130). Now, $10^{2n+1}=10^{2n} \times 10$, or $10^{2n+1}= 10^{2n} \times 10 + 10 - 10 = 10\,(10^{2n}-1) + 10$; adding 1 to both members, $10^{2n+1}+1 = 10\,(10^{2n}-1)+11$. But, according to (1) $10^{2n}-1$ is divisible by 11; moreover, 11 is divisible by itself. Hence, $10^{2n+1}+1$ is also divisible by 11.

This being established, let . . . . *hgfdcba* be the given number, which we will call N; we have $N=a+10b+10^2c+10^3d+10^4f\ldots$, an equation which we can put under the form

$$N=\left\{\begin{array}{lllll} & +(10+1)b & +(10^2-1)c & +(10^3+1)d & +(10^4-1)f\ldots \\ +a & -b & +c & -d & +f\ldots \end{array}\right\}$$

Now, according to the two preceding remarks, the first line is composed of numbers essentially divisible by 11, and forms, consequently, a first part, which is divisible by 11. Then, if the second part, which is nothing more *than the difference between the sum* a+c+f+h+ . . . . *of the figures in the odd places, and the sum of the figures* b+d+g+ . . . . *of the figures in the even places*, is divisible by 11, as we have supposed; the number, N, is also divisible by 11. Q. E. D.

133. When the difference between the sum of the figures in the odd places, and of those in the even places, is neither 0 nor a multiple of 11, the number itself is not divisible by 11, since one of its parts is divisible, and the other is not. But, then, there are two cases to be considered with reference to the manner of obtaining the remainder of the division.

1st. *If the sum of the figures of the odd places is greater than the second sum*, the difference is to be added to the first horizontal line of the value of N. Denoting then this first line by B, and the difference to be added by C, we will have, $N=B+C$; and if C is not divisible by 11, *the remainder of the division of* C *by* 11 *will be the same as that which we would obtain by dividing* N *by* 11 (129).

2d. *If, on the contrary, the sum of the figures of the odd orders is less than that of the figures of the even orders*, the difference will have to be subtracted from the first line, and we shall have $N=B-C$; C designating always the numerical value of the difference.

In order to determine in this case the remainder of the division of N by 11, let us observe that we have $B=11\times Q$, Q being an entire number, and $C = 11 \times Q' + R$; then, $N = 11 \times Q - 11 \times Q' - R$, or, subtracting and adding 11,

$$N = 11 \times Q - 11 \times Q' - 11 + 11 - R = 11(Q - Q' - 1) + 11 - R.$$

Whence we see in this case the remainder of the division of N by 11, is equal, *not to the remainder* R *of the division of* C *by* 11, *but to the difference between* R *and* 11.

In order to fix these ideas, let the number be 47356708. Adding up the figures in the odd places, we obtain (setting out from the right), 27; adding up the figures of the even orders, we obtain 13. Now, the first sum is greater than the second. Then, if we take the difference, which gives 14, the remainder, 3, of the division of this difference by 11, is equal to that of the division of the number itself. But, if we had the number 370546345, since the sum of the figures of the odd orders is 15, and that of figures of the even orders, 22, it follows that if we take the difference between the two sums, which gives 7, the remainder of the division of the number itself is not 7, but 11—7, or 4.

134. *Verification of multiplication and division, by the properties of* 9 *and* 11.

We cannot pass over a simple and very convenient means of verifying the multiplication and division of entire numbers. We enunciate this method as follows:

Add the figures of the multiplicand, and divide the sum by 9; *add the figures of the multiplier, and divide this sum also by* 9. *We thus obtain two remainders, which* (131), *are nothing more than the remainders of the division of these numbers by* 9. *Multiply these two remainders together, and divide their product by* 9; *this gives a third remainder. Finally, add the figures of the product, and divide the sum by* 9. *We obtain thus a fourth remainder, which is equal to the third when the multiplication has been accurate.* Let the two numbers be, for example, 5786 and 475, to be multiplied one by the other. The multiplication being performed, we add the figures of the multiplicand, rejecting the 9s by partial subtractions, as in (131). We thus obtain 8 for the first remainder. We operate in the same manner on the multiplier which gives 7 for remainder. This 7 we write under the 8, as in table. We then multiply 8 by 7, giving 56, which we divide by 9, giving 2 for remainder (or we can say 5 and 6 make 11, and 9 from 11 leave 2). Finally, we operate upon the product as upon the factors, which gives 2 for a fourth remainder. This being equal to the third, we conclude that the operation is exact.

$$
\begin{array}{rr|l}
5786 & 8 & 2 \\
475 & 7 & 2 \\
\hline
28930 & & \\
40502 & & \\
23144 & & \\
\hline
2748350 & &
\end{array}
$$

In order to establish this method of verification by 9 in a general manner, let us denote by A and B the two factors, by Q, Q′, R, and R′, the quotients and the remainders of the division of the multiplier and multiplicand by 9; we have the following equations,

$$A = 9 \times Q + R,$$
$$B = 9 \times Q' + R'.$$

Multiplying these two, member by member, we obtain

$$AB = 9 \times 9 \times Q \times Q' + 9 \times Q' \times R + 9 \times Q \times R' + R \times R'.$$

Now, the three first terms of the second number of this new equation, are evidently multiples of 9; then (129), the remainder of the division of the product AB by 9, must be that which the division of R × R′ by 9 gives. And this is what we wished to demonstrate. If one of the two factors of the multiplication is divisible by 9, the product ought to be so also; it is the same if the product R × R′ is divisible by 9. Or, we may express it thus: if one of the first remainders is 0, the third must also be 0. Hence, the fourth must be 0. Again, when the first two remainders are equal to 3, in which case the third remainder is equal to 0. Hence, the fourth must be 0. As to the verification of division, two cases can occur; either there will be a remainder after the ordinary operation is performed, or there will be none.

1st. If there is no remainder, the dividend is regarded as the product exact of the quotient and divisor; and we can apply the preceding rule regarding the divisor and quotient, as the two factors of a multiplication.

2d. If we obtain a remainder, we commence by subtracting this remainder from the dividend. The result of this subtraction will be the exact product of the quotient and divisor, and we operate upon these three as before.

N. B. *The verification by* 9 *is liable to several causes of error*, of which the following are the principal.

1st. It is possible that either in the partial products or in the total product, we may have written a 0 for a 9, or reciprocally; or, in the one, a figure *too small* or *too great* by a certain number of units, and in the other, a figure *too great* or *too small* by the same number of units.

2d. It is possible, also, when there are zeros in the multiplier, that we may not have written the partial products far enough to the left. We perceive at once, in these different cases, that the errors committed have no influence upon the *remainders* of the division by 9, of the terms of the operation to be verified.

The verification by 9 is only then, properly speaking, a half proof, to which we can have recourse when pressed for time; it being certain when the *third* and *fourth* remainders are not equal, that the operation is incorrect. But if they are equal, there is only a *great probability* that the product is the required one.

The verification by 11, which does not differ from that by 9, except in the manner of obtaining the remainder of the division of a number by 11, is preferable, though itself subject to some errors; but these errors occur much less often than in the method by 9. These verifications can be applied equally to the multiplication and division of decimal fractions, since these operations are performed in the same manner as in whole numbers.

135. There exist also, characteristics by which we can tell whether a number is divisible by the prime numbers, 7, 13, 17, ...; but the rules which it is necessary to follow, are longer in practice than the division of the number by 7, 13. . . . . .

These questions demand, moreover, a greater knowledge of algebra than the questions heretofore discussed.

We will, however, give the following question as an exercise for the pupil; *to determine in any system of numeration whatever, whose base is* b, *what numbers enjoy properties analogous to the properties of* 9 *and* 11 *in the decimal system, and to demonstrate these properties.* This can be solved very readily according to the principle, that in every system of numeration, any power whatever of the base can be expressed by unity, followed by as many zeros as there are units in the exponent of the power.

136. As to the *characteristics of the divisibility* of a number by the multiples, 6, 12, 15, 18, 36, 45, of the prime numbers, 2, 3, 5, they are sufficiently simple to find a place here.

1st. An even number is divisible by 6 or 18, when the sum of its figures is divisible by 3 or 9. For this number is then divisible by 2 and 3, or 9; now 2 and 3, 2 and 9, are prime with each other; then (127), the number is divisible by 6 or 18.

2d. A number is divisible by 12 or 36, when the two last figures form a number divisible by 4, the sum of the figures of the number being at the same time divisible by 3 or 9. For then, &c.

3d. Finally, a number is divisible by 15 or 45, when the last figure is 0 or 5, and in addition to this, the sum of the figures is divisible by 3 or 9.

We pass now to the method of finding all the divisors of a number, both *prime* and *multiple.*

137. We will divide this question into two distinct parts:

The *first* has for its object to determine *all the prime factors* which enter into any given number. and the *number of times* that *each prime factor enters.*

The *second* has for its object to obtain all the divisors, *prime* or *multiple,* which the number contains.

First Part. — *To decompose a number into all its prime factors.*

Let the number be for example, 2820.

$$\begin{array}{r|l} 2820 & 2 \\ 1410 & 2 \\ 705 & 3 \\ 235 & 5 \\ 47 & 47 \\ 1 & \end{array} \qquad 2820=2^2\times3\times5\times47.$$

We draw first a vertical line, to the left of which we place the number, the divisors to be written to the right of the same line: 2820 being divisible by 2, which we write opposite it on the right of the vertical line. We perform the division of 2820 by 2, and write the quotient, 1410, below the 2820. As 1410 is divisible by 2, we place this second divisor below the first; then the resulting quotient, 705, below the preceding, and we have

$$2820=2^2\times 705.$$

Now, we say, that the search for the prime divisors of 2820, other than 2, is now reduced to finding the prime divisors of 705. For, 1st. Every divisor of 705 must divide its multiple $2^2\times705$. 2d. Reciprocally, every prime divisor of 2820, other than 2, must divide 705.

We are then to operate upon 705 as upon the given number; 705 is divisible by 3; we write this new divisor under the preceding; then we place the corresponding quotient, 235, under the last already obtained, and from this results the new equality

$$2820=2^2\times 3\times 235.$$

235 not being divisible by 3, the question is reduced to finding the prime divisors of 235. Now, this number is divisible by 5, which we write in the column of divisors. The quotient of 235 by 5, 47, we place in the column of quotients.

We have then the equation

$$2820=2^2\times 3\times 5\times 47.$$

We are now led to seek the prime divisors of 47. But 47 is obviously itself prime; for the simplest prime number, after 5, is 7; and 7 will not divide it. Moreover, $7 \times 7 = 49$, a number greater than 47; whence we conclude that 47 is a prime number. Dividing it by itself, we set the quotient, 1, below the others. Here the operation ceases, and we have $2820 = 2^2 \times 3 \times 5 \times 47$ for the number 2820, decomposed into its prime factors.

138. *Important Remark.* — Before going farther, let us generalize what has just been said, in order to prove that 47 is a prime number; we will thus establish for every number *a limit* above which it is useless to go in the search for its prime divisors.

Let N be the given number, and suppose that we have tried in vain, as divisors, all the prime numbers up to ascertain number, a, the corresponding quotient of which is q, a fractional number less than a. We say, that the trial of any other number would be useless, and that N is a prime number

For we have, according to the supposition, $N = a \times q$ (q being fractional and less than a). Now, if there existed a number a^1 greater than a, which could exactly divide N, we would have, denoting the quotient by q',

$$N = a' \times q' \text{ (q' being an entire number).}$$

Whence, $a \times q = a' \times q'$. Now, a' being greater than a, q' must, to compensate, be less than q, which is itself less than a. Hence, the number N would have an entire divisor less than a, which is contrary to our hypothesis.

Take, for example, the number 263. No one of the prime numbers, 2, 3, 5, 11, 13, will divide this number. But, trying 17, we find a fractional quotient, $15 + \frac{8}{17}$, a number less than 17; whence, we conclude, that 263 is a prime number.

In general, the limit of the trials in the search for the prime divisors of a number, is the smallest prime *number* which gives a fractional number less than *this number* taken for the *divisor*. There are other limits which we will not investigate.

Let us now render general the method which we have, for the sake of clearness, commenced, by developing upon a particular example.

Let a be the smallest prime number, commencing with 2, which divides N. We divide N by a, the quotient by a, the second quotient by a, as long as the exact division is possible. Calling n the number of divisions which we have found it possible to perform, we have the equation $N = a^n \times N'$ (N′ being entire). We pursue the same course of reasoning as in (137), to show that the question is now reduced to operating upon N′ in the same manner; calling b the simplest prime number which divides N′, and p the number of successive divisions which can be performed, we have $N = a^n \times b^p \times N''$, (N″ being entire), admitting that c and d are the only factors of N″, so that we have

$$N'' = c^q \times N''', \text{ and } N'''d^s,$$

we obtain

$$N = a^n \times b^p \times c^q \times d^s,$$

and the number N is thus decomposed into its prime factors; and we know, too, the number of times that each one of these factors enters into it.

It results, moreover, from the general proposition (126), *that these prime factors, raised to the powers denoted by the exponents,* n, p, q, s, *respectively, form the only system of prime factors into which the number,* N, *can be decomposed.*

140. *Second Part.—To determine all the divisors, both simple and multiple, of any number whatever.*

From the same form under which we have just represented the number N, results a method of resolving this question. Let us write

$1, a, a^2, a^3, a^4, \ldots\ldots a^n$ ($n + 1$ terms).
$1, b, b^2, b^3, b^4, \ldots\ldots b^p$ ($p + 1$ terms).
$1, c, c^2, c^3, c^4, \ldots\ldots c^q$ ($q + 1$ terms).
$1, d, d^2, d^3, d^4, \ldots\ldots d^s$ ($s + 1$ terms).

It is evident that we would obtain all the divisors of N, unity included, by multiplying all the terms of the first line by all of the second, then all the terms of the product, by all the terms of the third line, and, finally, all the terms of the new product by those of the fourth line, since the different terms of this last product, are the products 1 and 1, 2 and 2, 3 and 3 . . ., of a, b, c, . . ., raised to powers whose degrees do not exceed $n, p, q,$ and s. Now, the number of this last product is $(n+1) \times (p+1) \times (q+1) \times (s+1)$. From this we deduce the following rule, also, for the total number of divisions of any number.

Increase by unity the exponents, n, p, q, s, . . . *of the different prime factors which enter into the number,* N. *Then multiply together these exponents, thus augmented by unity; the product expresses the total number of divisors of* N, *unity being comprised among the number.*

Let N, for example, be equal to

$$2^3 \times 3^2 \times 5^5 \times 7 \times 13^2.$$

The expression above becomes, in this case,

$$4 \times 3 \times 6 \times 2 \times 3, \text{ or } 432;$$

thus, the number N has 432 divisors.

141. The method which we have just indicated for determining all the divisors, prime and multiple, of a number, being not very convenient in practice, we will explain upon a new example, a more expeditious process.

	1
5880	2
2940	2, 4
1470	2, 8
735	3, 8, 12, 24
245	5, 10, 20, 40 | 15, 30, 60, 120
49	7, 14, 28, 56 | 21, 42, 84, 168 | 35, 70, 140 | 280 | 105, 210, 420, 840
7	7, 49, 98, 196 | 147, 294, 588, 1176 | 245, 490 980, 1960 | 735, 1470, 2940, 5880.
1	

In all, 4 × 2 × 2 × 3, or 48 divisors.

Explanation of the Table.

After having determined the prime divisors of 5880, by the method of (137), we write 1 above the factor 2, in the column of divisors. *We pass to the second divisor*, 2, by which we multiply the preceding; this gives the new divisor, 4, which we place on the right of the second divisor. Passing to the third divisor, 2, we multiply 4 only by 2, and place the product on the right of the third divisor. Passing to the divisor, 3, we multiply it by all the divisors which precede, viz: 2, 4, 8; which gives the new divisors, 6, 24, 48, which we place on the right of the divisor 3.

In a word, when we descend to a new divisor, we multiply all the divisors which precede by this divisor, taking care not to repeat, however, the products already obtained. It is certain that the products to which this mode of proceeding leads, comprehend all the divisors of the given number; since they are the combinations of the factors, 2, 3, 5, 7, raised respectively to powers whose exponents do not exceed 3 for 2, 1 for 3, 1 for 5, and 2 for 7.

142. The search for the prime factors of every number, is one of the most important questions of arithmetic, and one of the most useful in practice. One of the applications we have seen in finding the *least common multiple* of several numbers. We may also apply it in finding the *greatest common divisor* of two numbers, this being obviously *the product of all the prime factors common to the two numbers.* Thus, for example:

We find for the prime divisors of the number 2150, $1 \times 2 \times 5 \times 5 \times 43$, and for the number 3612, $1 \times 2 \times 2 \times 3 \times 7 \times 43$. Hence, the G. C. D. of these two numbers is $2 \times 43 = 86$. The reasons are obvious from the preceding articles.

FORMATION OF A TABLE OF PRIME NUMBERS.

143. The principles which we have established concerning *prime numbers,* and the application which have been made of

them, show sufficiently the utility of a *table* of this sort of numbers, as extended as possible.

There are several tables of this sort, some of them comprehending all the prime numbers from 1 to 3036000.

To give some idea of the manner in which such tables are made, suppose that we wished to form a table of prime numbers from 1 to 1000.

The first thousand numbers are written one after another in the most convenient form possible; for example, in ten columns, containing one hundred numbers each.

We then proceed as follows:

We draw lines across, 1st, all the even numbers except 2; 2d, all the multiples of 3 except 3, which remain after the first operation; 3d, and, in the same manner, the multiples of 5, other than 5, which have not been crossed in the first two operations. This done, we can affirm that all the numbers which have not been thus marked, from 1 to 7 × 7, or 49, are *prime numbers*, since all the multiples of 2, 3, and 5, as well as the multiplies of 7, below this limit, have necessarily been marked; and we have thus the prime numbers from 1 to 47. In the same manner, if we mark all the multiples of 7, from 49 up to 121, or 11 × 11 (11 being the prime number which comes directly after 7), we are then certain that the numbers preceding 121, which are not marked, are prime numbers; we thus obtain all the prime numbers from 47 to 113, inclusive.

Without carrying the details of this operation any farther, it is easy to see that we are thus led to suppress, successively, all the multiples not yet suppressed, of the prime numbers already found, 11, 13, 17, . . , until we arrive at the number 997, the last one remaining of the first thousand numbers, after the suppression already made of 998, 999, and 1000, as multiples of 2 and 3. We find, thus, the succession of 169 prime numbers comprised between 1 and 1000, the table of which we subjoin, adding the six prime numbers which follow them.

Table of Prime Numbers from 1 *to* 1033.

1
2
3
5
7
11
13
17
19
23
29
31
37
41
43
47
53
59
61
67
71
73
79
83
89
97
101
103
109
113
127
131
137
139
149
151
157
163
167
173
179
181
191
193
197
199
211
223
227
229
233
239
241
251
257
263
269
271
277
281
283
293
307
311
313
317
331
337
347
349
353
359
367
373
379
383
389
397
401
409
419
421
431
433
439
443
449
457
461
463
467
479
487
491
499
503
509
521
523
541
547
557
563
569
571
577
587
593
599
601
607
613
617
619
631
641
643
647
653
659
661
673
677
683
691
701
709
719
727
733
739
743
751
757
761
769
773
787
797
809
811
821
823
827
829
839
853
857
859
863
877
881
883
887
907
911
919
929
937
941
947
953
967
971
977
983
991
997
1009
1013
1019
1021
1031
1033

144. *Remark upon the greatest common divisor.*

We may find it necessary sometimes to find the greatest common divisor of several numbers. For this we give the following rule. *We find first the* G. C. D. *of two of the numbers, then the* G. C. D. *of the one already found and a third number, then the* G. C. D. *of this last common divisor and a fourth number. . . . . .*

Let A, B, C, E, F, . . . be the given numbers, and call D the G. C. D. of A and B, D′ the G. C. D. of C and D. Then we say that D′ is the G. C. D. of A, B and C. For the G. C. D. of A, B and C, must divide D, and moreover must divide C. Hence, the greatest number which divides both C and D, is the greatest common divisor of A, B and C, and D′ is that number. The same course of reasoning will apply to the rest of the numbers.

We see that there is some advantage in operating first upon the two simplest numbers, since the G. C. D. sought cannot exceed that which exists between these two numbers.

We could also decompose the numbers into their prime factors, and proceed as in the method proposed in (142).

145. *Remark upon the least common multiples.*

We have already given a method of finding the least common multiple of several numbers in the chapter on vulgar fractions, which is rendered complete by the method of obtaining the *prime factors of any number whatever*, given in (137, 138). We give here another method founded upon the preceding theories. We consider, first, the two numbers, A and B. Denoting their greatest common divisor by D, and by q, q' the quotients of the division of A and B by D, we have the two equations, $A = D \times q$, $B = D \times q'$; q and q' being prime with each other.

Now, we say, that the least common multiple required is equal to

$$D \times q \times q'.$$

For this product is obviously a multiple of A and B, since it is divisible by $D \times q$ and $D \times q'$; it remains to be proved that it is the *least multiple* which we can obtain.

Let us call *M* any multiple whatever of A and B. In order to be divisible by A or $D \times q$, M must contain all the factors which enter into each one of the numbers, D and q; for the same reason it must contain all the factors of each of the numbers D and q'; and since q and q' are prime with each other, M cannot be less than $D \times q \times q'$. We have then the following rule: Determine the G. C. D. of A and B; then divide A and B by the G. C. D.; multiply the G. C. D. and the product of the two quotients; this gives M for the least common multiple of A and B. Find the least common multiple in the same manner of M and C. This will be the L. C. M. for A, B, C. Operate in the same manner on all the rest of the numbers in succession. The method given under the head of reducing fractions to the least common denominator, can be reduced to practice thus, (now that we know the method of finding all the prime factors of any number).

For example, take the numbers, 6, 9, 4, 14, and 16.

2)6, 9, 4, 14, 16
2)3 9 2 7 8
3)3 9 1 7 4
1 3 1 7 4 2×2×3×3×7×4=least com. mult.

We place the numbers in a horizontal line, and commence with the prime number, 2, as a divisor. We divide all those numbers which are divisible by 2, and bring down the quotients, together with the numbers not divisible. We proceed in the same manner with the quotients, until there are no two which 2 will divide. We then divide the last quotients and numbers brought down by the prime number, 3, and continue the operation until there are no two numbers left divisible by any number greater than unity. We then multiply the divisors and the numbers thus remaining together for the least common multiple. It is evident that we thus form the least number divisible by the given numbers.

Of Periodical Decimal Fractions.

146. The valuation of vulgar fractions by decimals, that is to say, by *tenths, hundredths* . . . . of the principal unit, gives rise to singular circumstances which merit an examination. But, before entering upon the discussion of them, we must return to the method for converting a vulgar fraction into a decimal.

We have seen that, in order to effect this reduction, we must

1st. *Annex a* 0 *to the numerator, and divide the resulting number by the denominator; this gives the tenths of the quotient and a remainder.* 2d. *Write a new* 0 *on the right of the remainder, and divide by the denominator, obtaining thus the hundredths of the quotient.* We continue this operation until we have reached the degree of approximation required. This process is evidently the same as *multiplying the numerator by unity, followed by as many zeros as we wish decimal figures in the result; then dividing the result by the denominator, and pointing off in the quotient the number of decimal figures required.*

147. This enables us to demonstrate the two following properties:

1st. *Every vulgar fraction whose denominator does not contain any prime factors other than* 2 *and* 5, *is reducible to a limited number of decimal figures; that is to say, after a certain number of operations,* we must arrive at a remainder equal to 0; in which case the decimal fraction obtained expresses the exact value of the given fraction. Besides, *if the fraction is reduced to its simplest form, the total number of operations to be performed in order to reduce it to its equivalent decimal is always equal to the greatest of the two exponents of* 2 *and* 5, *which enter into the denominator.*

Thus, the fractions

$$\tfrac{7}{8},\ \tfrac{13}{25},\ \tfrac{11}{40},\ \tfrac{317}{1250},$$

which can be placed under the forms

$$\frac{7}{2^3},\quad \frac{13}{5^2},\quad \frac{11}{2^3 . 5},\quad \frac{317}{2 . 5^4},$$

are reducible to a limited number of decimal figures. The first gives rise to three operations, the second to *two*, the third to three, and the fourth to 4.

We find, in fact, for their values,

0·875; 0·52; 0·275; 0·2536.

In order to prove this property generally, we remark, that 10, 100, 1000 . . . . being equal to 2×5, $2^2\times 5^2$, $2^3\times 5^3$, ..., if, in order to effect the reduction to a decimal fraction, we multiply the numerator by 10, 100, 1000 . . . ., the resulting product will necessarily be divisible by 2×5, $2^2\times 5^2$ . . . .; then, in multiplying this numerator by unity, followed by as many zeros as there are units in the greatest of the exponents of 2 and 5, which the denominator contains, the resulting product will necessarily be a *multiple* of this denominator.

Then, the number of operations to be performed is equal to the greatest of the two exponents of 2 and 5, which enter the denominator of the given vulgar fraction.

148. *Every irreducible vulgar fraction, whose denominator contains one or more prime factors different from* 2 *and* 5, *gives rise to an infinite number of decimal figures. Moreover, the decimal fraction resulting from it is periodical;* that is to say, after a certain number of operations, the same decimal figures recur again.

For the multiplication of the numerator by 10, 100, 1000, can only cause the introduction of the two factors, 2 and 5, raised to certain powers; thus, the prime factor which we suppose to be in the denominator, and not in the numerator, will not be in the latter, after this multiplication by 10, 100, ... Then, whatever number of zeros we add, we shall never obtain a product exactly divisible by the denominator; thus, the operations can be carried on to infinity.

We say, moreover, that the decimal fraction will be *periodical.* For, as each remainder is always less than the divisor, it follows that when we shall have performed as many divisions as there are units less one in the divisor, we will necessarily arrive at a remainder already obtained, (if, in fact, this remainder does not recur sooner). Now, annexing a 0 to this remainder, we will have a partial dividend exactly the same with one of the preceding; whence it follows, that we will have a series of quotients and remainders equal to the *preceding, recurring periodically,* setting out from the first partial dividend, which is equal to any of the preceding.

Let us make some applications of this.

149. Required to reduce the fraction, $\frac{6}{7}$, to decimals.

$$\begin{array}{l} 60)7 \\ \overline{40}(\overline{0{\cdot}857142 \mid 857142} \\ \quad\overline{50} \\ \quad\ \overline{10} \\ \quad\ \ \overline{30} \\ \quad\ \ \ \overline{20} \\ \quad\ \ \ \ \overline{6} \end{array}$$

Here the period shows itself after the 6th partial division.

Second Example.—Let the fraction be $\frac{13}{37}$.

$$\begin{array}{l} 130)\ 37 \\ \overline{190}(\overline{0{\cdot}331 \mid 331} \\ \quad\overline{50} \\ \quad\ \overline{13} \end{array}$$

In this example, the period commences with the fourth partial division.

Third Example, $\frac{29}{84}$.

$$\begin{array}{l} 290)\,84 \\ \overline{380}\,(\overline{0{\cdot}34523809 \mid 523809} \\ \quad\overline{440} \\ \quad\ \overline{200} \\ \quad\ \ \overline{328} \\ \quad\ \ \ \overline{680} \\ \quad\ \ \ \ \overline{800} \\ \quad\ \ \ \ \ \overline{44} \end{array}$$

The period is manifest here after the 8th operation. But the two first decimal figures form no part of the period, while in the first two examples the period commences with the first decimal figure. The periodical decimal fractions, whose period commences with the first decimal figure, are called *simple periodical fractions;* and those whose period commences after a certain number of decimal places already written, are *mixed periodical fractions.*

150. We have just seen that certain vulgar fractions, reduced to decimals, give rise to periodical decimal fractions.

Reciprocally, every periodical decimal fraction, simple or mixed, arises from a vulgar fraction, which can easily be found from any given periodical fraction.

This question presents two distinct cases; either the periodical fraction is *simple* or it is *mixed.* Let us consider the first case.

Take, for example, the periodical fraction

$$0{\cdot}513513513513 \ldots .$$

and let us designate by N the fraction which has given rise to it.

We have $N=0{\cdot}513513513 \ldots . \quad (1)$

Multiplying the two members of this equation by 1000, which is done in the second member by removing the point three places towards the right, we obtain

$$N\times1000=513{\cdot}513513513 \ldots .$$

Or (2) $\quad N\times1000=513+0{\cdot}513513513 \ldots .$

Subtracting (1) from (2) we have

$$N\times999=513.$$

$$\text{Then } N=\tfrac{513}{999}.$$

Let the fraction be $N=0{\cdot}714285714285 \ldots . \quad (1)$

Multiply both members by 1000000, we have

$$N\times1000000=714285{\cdot}714285 \quad (2)$$

Subtracting the first from the second,

$$N \times 999999 = 714285$$
$$N = \frac{714285}{999999}$$

Reducing the fractions $\frac{513}{999}$ and $\frac{714285}{999999}$ to their simplest terms, we get

$$\frac{19}{37} \text{ and } \frac{5}{7}.$$

What we have shown proves *that a simple periodical fraction is equivalent to a vulgar fraction which has for numerator the figures of the period, and for denominator a number composed of as many* 9*'s as there are figures in the period.*

Thus, for an additional example, the fraction 0·351351351 . . . . is equivalent to the fraction $\frac{351}{999} = \frac{39}{111} = \frac{13}{37}$.

Again, the fraction 0·03960396 . . . . is equivalent to $\frac{0396}{9999}$, or simply $\frac{396}{9999} = \frac{44}{1111} = \frac{4}{101}$.

In general, if $x = 0$, *abcde* . . . . *abcde* . . . . *abcde* (where *abcde* . . . . represent decimal figures with their relative values and *not products*), we shall have

$$x = \frac{abcde \ldots.}{99999 \ldots.}$$

N. B. If the periodical fraction contains an entire part, we do not regard it in forming the vulgar fraction; but we add it to the vulgar fraction found after it is reduced to its simplest terms.

Thus, given the periodical fraction

$$4{\cdot}162162 \ldots.$$

We have, first,

$$0{\cdot}162162 \ldots. = \frac{162}{999} = \frac{18}{111} = \frac{6}{37}.$$

Then, $$4{\cdot}162162 = 4 + \frac{6}{37} = \frac{154}{37}.$$

151. *Second Case.*—Required to find the equivalent vulgar fraction, or *generatrix*, as it is sometimes called, of a *periodical mixed fraction.*

Given, for example, the fraction

$$3{\cdot}45891891\ldots.$$

Multiplying this fraction by 100, we obtain 345·891891; and, according to (N. B.) of preceding article, this expression has for its value

$$345+\frac{891}{999},\ \text{or}\ \frac{345\times 999+891}{999},$$

$$\text{or},\ \frac{345\times(1000-1)+891}{999},\ \text{or}\ \frac{345546}{999}.$$

But, as we have multiplied the fraction by 100, in order to reduce the result to its true value, we must divide it by 100; we thus obtain $\frac{345546}{99900}$, a fraction which, reduced to its simplest form, becomes $\frac{6399}{1850}$, the generatrix of the *mixed periodical fraction*, 3·45891891 . . . .

If the fraction were under the general form,

$$0,pqrs,abcde,abcde\ldots.$$

its value would be

$$pqrs+\frac{abcde}{99999},$$

after multiplying it by 10000, or

$$\frac{pqrs\times 99999+abcde}{99999},$$

or, reducing the result to its true value,

$$\frac{pqrs\times 99999+abcde}{999990000}.$$

We say, then, *any mixed periodical fraction whatever is equivalent to a vulgar fraction which has for its numerator the period, augmented by the product of the part which precedes the period by a number composed of as many* 9s *as there are figures in the period, and for denominator this same number of* 9s, *followed by as many zeros as there are figures in the part which precedes the period.*

Take, for another example,

0·3193069306.

The preceding rule gives for its value,

$$\frac{9306 + 31 \times 9999}{999900} = \frac{9306 + 31\,(10000 - 1)}{999900} =$$

$$\frac{309969 + 9306}{999900} = \frac{319275}{999900} = \frac{129}{404}.$$

We give here as examples of simple and mixed periodical decimals,

1st. 0·9999 . . . . $= \frac{9}{9} = 1$
2d. 0·012345679012345679 . . . . $= \frac{1}{81}$
3d. 0·987654320987654320 . . . . $= \frac{80}{81}$
4th. 16·285714285714 . . . . =
5th. 4·9428571428571 =
6th. 5·52027027 =

152. The expression $\dfrac{pqrs \times 99999 + abcde}{999990000}$ leads to some remarkable consequences. It can be put under the form

$$\frac{pqrs\,(100000 - 1) + abcde}{999990000},$$

equal to

$$\frac{pqrs00000 - pqrs + abcde}{999990000}$$

This being established, it is obvious from this last form, that if the calculations which are indicated in the numerator are effected, the result cannot be terminated by one or more zeros; for, in order that this should be the case, it would be necessary that some of the last figures of *pqrs* should be the same as the last figures of *abcde;* and, in this case, the period would not commence after the 4th decimal figure, as we have supposed. (For example, if we had $s = e$, $r = d$, the primitive fraction would be 0, *pqdeabcdeabc* . . . .) We see, then, that after the reduction of the expression above to its simplest terms, the result must be a fraction, whose denominator contains the two factors, 2 and 5, or at least one of the two, to the 4th power; that is to say, to a power whose degree is denoted by the number of figures which form no part of the period.

We can infer from this the two following propositions:

1st. *Every fraction whose denominator does not contain either of the two factors,* 2 *and* 5, *or is prime with* 2 *and* 5, *gives rise, when reduced to decimals, to a simple periodical fraction.*

For, if we could obtain a mixed periodical fraction, it should follow, that the equivalent vulgar fraction, which we obtain by the rule in (151), being reduced to its simplest terms, should be equal to the given fraction. Now, that is impossible, (for in order that one irreducible fraction be equal to another fraction, the terms of this last must be the same multiples of the terms of the first).*

It results, then, that the denominator of the proposed fraction would be a multiple of 2 or of 5; which is contrary to the hypothesis.

2d. *Every irreducible fraction, whose denominator contains one of the factors,* 2 *and* 5, *or both, raised to a certain power, gives rise to a mixed periodical fraction, whose period must commence after we have found as many decimal figures as there are units in the greater of the two exponents of* 2 *and* 5, *which enter into the denominator.*

First, the periodical fraction cannot be simple; for the formula for these sorts of fractions being $\frac{abcde\,\ldots\,.}{99999\,\ldots\,.}$, it is impossible that this fraction, whose denominator does not contain either of

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* The terms of every irreducible fraction are prime with each other, and every fraction whose terms are prime with each other is an irreducible fraction. This is obvious, as this reduction depends upon suppressing the common divisor of the two terms. Hence, it is obvious, that no two irreducible fractions can be equal, unless the terms are identical in both, nor can an irreducible fraction be equal to any other fraction whose terms are not the same *multiples of the terms of the first fraction.*

the factors, 2 and 5, should be equal to the given fraction whose denominator contains these factors.

In the second place, the period must commence after n figures, if n express the greater of the two exponents of 2 and 5, which is found in the denominator; for suppose, for example, that it commences after $n-1$ figures; the equivalent to this periodical fraction would have a denominator which would only contain the two factors, 2 and 5, or one of them to the $(n-1)$th power, and could not be equal to the given fraction, since these two fractions are supposed to be irreducible.

For example, the fractions $\frac{6}{7}$, $\frac{13}{37}$, (149), gave simple periodical fractions, because 7 and 37 are prime with 2 and 5; but the fraction, $\frac{29}{84}$, gave a mixed periodical fraction, whose period commences after the second figure, because 84 is equal to $2^2 \times 21$.

Finally, the fraction, $\frac{145}{176}$, which can be put under the form $\frac{145}{2^4 . 11}$, should give a periodical fraction whose period commences after the 4th decimal figure.

We find, in fact, for the value of this fraction in decimals,

$$0{\cdot}8238636636 \ldots .$$

153. We will not carry farther the examination of the properties of periodical decimal fractions, but close by observing that properties analogous to the preceding manifest themselves in any system of numeration whatever. The fractions in any other system, which enjoy these analogous properties, are those whose denominators are powers of the *Base* of the system. Let this *base* be b.

First, in order to reduce a vulgar fraction into subdivisions b times smaller than unity, and into other subdivisions b times smaller than the first, &c., it would be necessary to multiply the numerator by b or 10; that is to say, to annex a 0, and divide the result by the denominator; which should give in the quotient units b times smaller than the principal unit, and a certain remainder; to write a new 0 on the right of the remainder, and divide the result by the denominator, giving in the quotient units b times smaller than the preceding, and b^2 times smaller than the principal unit, and so on. This being established, we deduce from it by reasoning precisely the same as that which served to establish the properties of decimal fractions which arise from vulgar fractions, that *the vulgar fractions in a system whose base is* b, *being converted into subdivisions* b, b^2, *&c., smaller than unity, give rise to fractions (analogous to decimals) of a limited or infinite number of figures, simple or mixed periodical, and that the composition of the denominator of the vulgar fraction with reference to the prime factors which enter into the base* b, *suffices to characterize these different sorts of fractions.*

We propose as an exercise for the pupil the investigation of the enunciations and demonstrations of these properties.

EXERCISES.

1. Prove that every entire even number is the sum of several powers of 2, and that every entire odd number is the sum of several powers of 2, augmented by unity.

Examples, 876, 2539, 6750.

2. Every entire number, which is not prime, has at least one prime divisor other than unity.

3. The remainder of the division by 9 of the product of any number of factors, is equal to the remainder which the product of the remainders of the division of each factor by 9 gives.

Prove that this property belongs to every number, and not to 9 alone.

4. The product of any three entire consecutive numbers is always divisible by 6.

5. Convert the numbers, 345 and 225, of the decimal system, into their equivalents in the binary system. Add these last in the binary system, and convert the *sum* back to the decimal system.

6. All the prime numbers, except 2 and 3, augmented or diminished by unity, are divisible by 6; that is, they are comprised in the general formula, $6n \pm 1$, (read plus or minus), n being any entire number.

7. If the sum of the figures of any number be subtracted from the number itself, the remainder will be divisible by 9.

8. The expression $n(n+1)(2n+1)$ is always divisible by 6.

CHAPTER VI.

APPLICATION OF THE RULES OF ARITHMETIC.—THEORY OF RATIOS AND PROPORTION.

154. Introduction. — We have seen, in the course of the explanation of the different operations of arithmetic, that these operations give rise to two principal species of questions. 1st. Those which have for their object to demonstrate the existence of certain properties of certain numbers known and given. 2d. Those in which it is proposed to find certain numbers from the knowledge of other numbers having fixed relations with the first. The first are *theorems*, properly speaking; but we have generally called them *Principles* and *Propositions*. The questions of the second species, which are not particular applications of the rules and principles, are called *Problems*.

The Problems, which we have hitherto solved, have been easy of solution, because the data were simple, and the relations between the known and unknown quantities very obvious. But this is not generally the case; as very often, in order to arrive at a solution, we have a considerable difficulty to overcome, which consists in discovering and determining the series of operations to be executed upon the numbers known and given, in order to arrive at a knowledge of the numbers sought.

Nevertheless, there exists a certain class of questions, the resolution of which can be subjected to *fixed and certain rules*; these are particularly those in which we consider *Proportional Magnitudes.*

The greater part of these questions are precisely those which the general necessities of society give rise to, in that which relates to its commercial, industrial, and financial interests; they are generally known as the *Rule of Three*, the Rules for the calculation of *Interest*, *Discount*, the Rule of *Fellowship*, *Exchange*, &c.

To arrive easily at the solution of these questions, we will commence by explaining the theory of ratios and proportions.

§ I. — Of Ratios and Proportions, and of their Principal Properties.

155. We have already said (1), that in order to form an idea of any magnitude whatever, we must compare it with some other magnitude agreed upon, of the same species, which can be taken arbitrarily or in nature. The result of this comparison is what we have called *number*. Number, then, expresses the *relation* between any magnitude and its unit. Now, if we wish to compare any two magnitudes whatever, of the same species, or what is the same thing, to compare the numbers which express them, the result of this comparison is a *relation* between these two numbers.

When we thus compare two magnitudes with each other, we may either wish to know *how much* the greater exceeds the less, or *how many times* the greater contains the less. From this results two sorts of relations between the numbers compared, one which is sometimes called an *Arithmetical ratio*, and another called a *Geometrical ratio*. But these names, which are but little significant, are well replaced by the word *difference*, in order to express the result of the comparison by subtraction, and *Ratio* to express the result of the comparison by division.

Thus, let 24 and 6 be the two numbers which we wish to compare. We have $24-6=18$ for the *difference*, and $\frac{24}{6}=4$ for the Ratio.

The relations of magnitudes by division or Ratios will chiefly occupy the present chapter, as by far the most important of the two classes of relations; but we will first give one or two leading properties of the *Relations by Subtraction* or Differences.

156. In every Difference or Ratio, the two terms are thus distinguished. The one first written is the antecedent; the second term is the consequent. Thus, in the expressions $24-6$, $\frac{24}{6}$, 24 is the *antecedent* in both cases, and 6 is the *consequent.* When the *difference* between two numbers is equal to the *difference* between two other numbers, the four numbers taken together form an *Equi-difference.*

For example, let the four numbers, 12, 5, 24, 17; the difference of 12 and 5 is 7; the difference of 24 and 17 is also 7. These, then, form an equi-difference which we write thus:

$$12.5 : 24.17.$$

Placing *one* point between 1st and 2d terms, *two* points between 2d and 3d, and one between the 3d and 4th. We enunciate it

12 is to 5 as 24 is to 17;

that is, 12 exceeds 5 by as many units as 24 exceeds 17. We can also write it

$$12-5=24-17;$$

12 and 24 are the *antecedents;* 5 and 17 the *consequents.* The first and last term are moreover called the *extremes;* the second and third the *means.*

This established, *we say that, in every equi-difference, the sum of the extremes equals the sum of the means.*

Let $11.7 : 19.15$;

We have obviously $11+15=7+19$.

To prove this generally, we observe that if the consequents were equal to their antecedents, as for example,

$$11.11 : 19.19,$$

the proposition would be manifestly true. Now, in order to place the first equi-difference under this form, we have simply to add 4 to each of its consequents; that is, the sum of the means and sum of the extremes are augmented by the same number. Hence, if these sums are equal now, they must have been so before. Then, &c.

As a consequence of this property, knowing three terms of an equi-difference, we can find the fourth. Thus, let

$$23.11 : 49.x, \text{ (x being the unknown term)},$$

be the equi-difference, we have

$$23 + x = 49 + 11;$$

whence x is known. Sometimes two of the terms of the equi-difference are the same as

$$27.39 : 39.51.$$

Here the double of one of the means is equal to the sum of the two extremes, or the mean itself is equal to half the sum of the extremes. Thus, in the equi-difference,

$$23.x : x.49,$$

$$x = \frac{49 + 23}{2} = 36,$$

and this number is called the *average* or *arithmetical mean* of the two numbers.

It is useless to proceed farther with the properties of equi-differences, as they are of very little use. We will merely add, that no transformation executed upon an equi-difference destroys this equi-difference, so long as the sum of the extremes remains equal to the sum of the means.

We pass to the discussion of Ratios and Proportions, properly so called.

157. The *ratio* of two magnitudes, we have seen, is the quotient of the division of the numbers which express these magnitudes. This ratio can be an entire number or a fractional

number, greater or less than unity. For example, the ratio of 24 to 6 is $\frac{24}{6}$, or 4; that of 6 to 24 is $\frac{6}{24}$, or $\frac{1}{4}$; that of 75 to 18 is $\frac{75}{18}$, or $\frac{25}{6}$.

It is in the sense of *Ratio* that we have hitherto understood the comparison of any magnitude whatever with its *unit* (No. 1). In the theory of compound numbers, the *relation* of the principal unit to its subdivisions, or between two subdivisions, is the number of times which the one contains the other.

158. The comparison of two concrete numbers supposes always that these magnitudes are of the same species, since we cannot compare magnitudes of different species (No. 2).

The ratio is itself, by its very definition, essentially an *abstract number*, expressing how many times one of the numbers contains the other, or is contained in it. The antecedent and consequent, which form the ratio, are, we have seen, the numerator and denominator of a fractional expression which we obtain, in indicating the division of the two magnitudes which we are comparing.

159. When the ratio of two numbers is equal to the ratio of two other numbers, we say, that the four numbers or magnitudes which they represent, are in *proportion*, or *proportional*. A *proportion* is then the expression of the equality of two *ratios*.

For example, the ratio of 48 to 12 being 4, and of 36 to 9 being also 4, we have the equation

$$\frac{48}{12} = \frac{36}{9}, \text{ or } 48 : 12 = 36 : 9.$$

It is sometimes more convenient to present the proportion under the form 48 : 12 : : 36 : 9, which is thus enunciated:

48 is to 12 as 36 is to 9.

The terms 48 and 36 are antecedents; 12 and 9 are consequents. The first and fourth are *extremes;* the second and third are *means.*

160. *Fundamental Property.* — All proportions possess a property which may serve as a basis for the resolution of the problems whose enunciations contain *proportional* quantities. This property consists in this :

In every proportion the product of the extremes is equal to that of the means.

Let the proportion be

$$(1) \quad 24 : 18 : : 20 : 15,$$

of which the ratios $\frac{24}{18}$ and $\frac{20}{15}$ each equals $\frac{4}{3}$. We say, that we must have

$$24 \times 15 = 18 \times 20.$$

For the property would be evident if we had the proportion

$$24 : 24 : : 20 : 20, \quad (2)$$

(which we call an identical proportion). Now, to render the proportion (1) the same as (2), it suffices obviously to multiply each consequent by $\frac{4}{3}$; but by this, we multiply the product of the extremes and the product of the means by the same number, and make the same change in both. Hence, if equal after the multiplication, they must have been equal at first. Hence the property is proved.

161. *Reciprocally.* — *If the product of two numbers is equal to the product of two other numbers, these four numbers form a proportion of which either pair of factors will constitute the means, the other pair constituting the extremes.*

For, if no proportion existed among these four numbers, it would be necessary, in order to render the second and fourth respectively equal to the first and third, to multiply each one by a *different* number, expressing in the one case the ratio of the first term to the second, in the other of the third to the fourth; and as the two products would thus become equal by the multiplication of each by a *different number*, it would result that they were not equal before the multiplication; which would be contrary to the enunciation of the proposition.

Then, &c., &c.

162. Another demonstration of the fundamental property and its reciprocal. (We employ letters in order to render the reasoning more concise and general).

Let a, b, c, d, be four numbers in proportion, so as to give

$$a : b :: c : d, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

If we multiply the two members of this equality by $b \times d$, product of the two consequents, we obtain

$$\frac{a \times b \times d}{b} = \frac{c \times b \times d}{d}.$$

Suppressing in each member the factor common to the numerator and denominator, we have

$$a \times d = c \times b.$$

Then the product of the extremes is equal to that of the means.

Reciprocally, let the four numbers, a, b, c, d, be such, that we have

$$a \times d = b \times c.$$

Let us divide the two members of this equality by $b \times d$, product of one factor of the first member by one factor of the second, we have thus

$$\frac{a \times d}{b \times d} = \frac{b \times c}{b \times d};$$

or, simplifying,

$$\frac{a}{b} = \frac{c}{d}, \text{ or } a : b :: c : d.$$

Thus, *the four numbers form a proportion of which the factors of the first product constitute the extremes, the factors of the second product the means.*

163. *First Consequence.—In every proportion we can cause,* 1st, *the two means to exchange places;* 2d, *the two extremes to change places;* 3d, *the means to exchange places with the extremes without destroying the proportion between the four numbers thus written.*

For it is evident that these changes do not alter the equality of the two products which the extremes and means of the primitive proportion give. And since, in the new expressions, the product of the *first number* by the *last* always remains equal to the product of the *second* by the *third*, there will always exist a proportion between the four numbers after the changes are effected.

Let the proportion be, for example,

$$48 : 36 : : 72 : 54. \quad (1)$$

We have, by changing the means for each other,

$$48 : 72 : : 36 : 54. \quad (2)$$

By exchanging the extremes,

$$54 : 36 : : 72 : 48. \quad (3)$$

By placing extremes in the places of the means, and the means in the places of the extremes,

$$36 : 48 : : 54 : 72. \quad (4)$$

In the expressions (2), (3), (4), the product of the second number by the third, is

$$36 \times 72, \text{ or } 48 \times 54;$$

and the product of the first by the fourth,

$$48 \times 54, \text{ or } 36 \times 72.$$

Now, these products are equal by virtue of proportion (1); then the expressions (2), (3), and (4), are also proportions.

The common ratio of (1) is $\frac{4}{3}$, of (2), $\frac{2}{3}$, of (3), $\frac{3}{2}$, and $\frac{3}{4}$ for the proportion (4).

N. B. It is obvious that inverting the order of the terms in each ratio does not destroy the proportion, since it amounts to the same change as is exhibited in (4).

164. *Second Consequence.* — *We can in every proportion* multiply or divide *one extreme* and *one mean* by the *same number, without destroying the proportion.*

For the products of the extremes and means of the given proportion being equal, the new products which result from the multiplication or division of these products by the same number will also be equal; and the proportion will still exist. There are many other properties of proportions; but those which we have just developed are the only ones of which we shall have need for the resolution of the problems which depend on this theory.

§ II. — RESOLUTION OF QUESTIONS DEPENDENT ON THE THEORY OF PROPORTIONAL QUANTITIES.

Rule of Three.

165. A great number of problems in commerce, banking, &c., contain in their enunciation numbers bearing relations to each other susceptible of being expressed by proportions. Of these numbers some are given and known, the others unknown, to be determined. We designate, under the title, the *Rule of Three*, the process by which *we find the fourth term of a proportion when three terms are given.*

Now, from the property of every proportion that the product of the extremes is equal to the product of the means, it results necessarily that, in order to obtain the value of the unknown term, *we must, if it is an extreme, divide the product of the means by the known extreme.*

And if it is a mean, we must divide the product of the extremes by the known mean.

Thus, let the two proportions be

$$24 : 9 : : 32 : x; \; 45 : 36 : : x : 24;$$

(we denote the unknowns by the last letters of the alphabet).

Since the first gives $24 \times x = 9 \times 32$, there results

$$x = \frac{9 \times 32}{24} = 12;$$

we have also for the second,

$$36 \times x = 45 \times 24.$$

$$\text{Whence, } x = \frac{45 \times 24}{36} = \frac{1080}{36} = 30.$$

The proportions become then

$$24 : 9 :: 32 : 12;\ 45 : 36 :: 30 : 24.$$

The common ratio is $\frac{8}{3}$ for the first, and $\frac{5}{4}$ for the second.

We pass now to the resolution of some problems, of which those in (41) may be considered particular examples.

166. *Problem First.* — *Required, the price of* 384 *lbs. of a certain commodity,* 25 *lbs. of which cost* \$6·50?

Analysis. — Since 25 *lbs.* cost \$6·50, it is clear that 2, 3, 4 . . . . times 25 *lbs.* must cost 2, 3, 4 . . . . times as much; thus, the two given numbers of pounds bear to each other the same relation as their respective prices. Then, if we designate by x the unknown price of 384 *lbs.*, and if we consider for the moment the three given numbers and x as abstract numbers, we have the proportion

$$(1) \quad 25 : 384 :: 650 : x.$$

$$\text{Whence (165), } x = \frac{384 \times 650}{25} = \frac{249600}{25} = 9984;$$

and we conclude that the 384 *lbs.* of the commodity ought to cost \$99·84.

N. B. Before seeking the value of x by means of the proportion (1), we can simplify that proportion in observing that the antecedents, that is, one extreme and one mean, are divisible by 25. We then suppress this factor (164), and obtain

$$1 : 384 :: 26 : x; \text{ whence, } x = 384 \times 26 = 9984.$$

Another method of resolution. — If 25 *lbs.* cost \$6·50, one pound must cost 25 times less, or $\frac{1}{25}$ of \$6·50; that is, $\frac{\$6{\cdot}50}{25}$.

Then, 384 *lbs.* will cost 384 times as much as 1 *lb.*, or $\frac{\$6{\cdot}50}{25} \times 384$; which gives \$99·84.

Second Problem. — *It takes* 135 *men* 20 *days to do a certain piece of work; how many days would* 300 *men require to perform the same labour?*

Analysis. — If a certain number of men have employed 20 days in accomplishing the work, it is clear that a number of men 2, 3, 4 . . . . times as great must occupy 2, 3, 4 . . . . times shorter period to do the same work, other things being equal; then, as many times as the first number of men, 135, is contained in the second number, 300, so many times the number of days necessary for the second number of men, or the number sought, x, will be contained in the number of days necessary for the first number of men.

Thus, we have the proportion

$$135 : 300 :: x : 20;$$

$$\text{whence, (165), } x = \frac{135 \times 20}{300} = 9.$$

Then, it takes 300 men 9 days to do the work. We could have suppressed in this proportion the factor, 15, common to the two first terms, and the factor, 20, common to the two consequents.

We should then have

$$1 : 9 :: 1 : x; \text{ whence, } x = 9.$$

Another mode of resolution. — If 135 men took 20 days to do the work, it would have taken one man 135 times as much time, or 135 × 20 days, and 300 men would have required a number of days 300 times as small as 20 × 135; that is to say,

$$\frac{20 \times 135}{300} = \frac{2700}{300} = 9 \text{ days.}$$

Ratios, Direct and Inverse.

167. Before treating more complicated problems, we must make known certain terms which the consideration of proportional quantities give rise to.

In every question, the enunciation of which contains four numbers in proportion, two of these numbers are of a certain species, and the two others of another species; but each term of the second species is closely connected by the conditions of the question with one of the terms of the first.

It is thus in the first problem (166), two of the four numbers express the weights of a certain commodity; the other two, the respective prices of these weights.

In the same manner, in the second problem, we had two numbers of *men*, and two numbers of *days;* and the latter expressed the respective periods employed by the two numbers of men to do the same work. It is agreed, for this reason, to call the two terms of different species, thus connected by the enunciation of the question, *Correspondents.*

For example, in the first problem, the prices are the *correspondents* of the pounds; and *vice versâ,* the numbers of pounds are the *correspondents* of the prices.

This established, we say, that there is a *Direct Relation* between the numbers of the first species and the numbers of the second; or that these numbers are *directly proportional,* when, the proportion having been established, we see that as each number *increases* or *diminishes,* its correspondent *increases* or *diminishes;* and that, on the contrary, the *Relation is Inverse,* or the four numbers are inversely (or reciprocally) proportional, when, as each number *increases* or *diminishes,* its correspondent *diminishes* or *increases.*

The enunciation of the first problem offers the example of a *direct relation;* for the greater the number of pounds, the greater the price.

The second problem gives rise to an *inverse relation;* for the more men there are to do the work, the shorter period required.

If the relation is *direct,* and if we wish to write the proportion under the form

$$a : b :: c : d,$$

one of the numbers, and its *correspondent,* must form the two *antecedents,* and the other two the *consequents.* On the contrary, if the relation is inverse, one of the numbers and its correspondent must form the extremes, while the other two form the two means. When we write the proportion under its equivalent form of two equal fractions,

$$\frac{a}{b} = \frac{c}{d},$$

it is necessary, in the case of the direct relation, that one of the numbers and its correspondent form the *two terms* of the first fraction, or the *numerators* of the two fractions, while the other numbers form the *two terms* of the second fraction, or the *denominators* of the two fractions; and, in the case of the inverse relation, each number and its correspondent must form the *numerator* of the *first* fraction and the *denominator* of the *second,* or the *denominator* of the *first* fraction and the *numerator* of the *second.*

N. B. All these distinctions in the manner of writing the proportions furnished by the enunciations of the problems are of importance, and should be carefully retained in the memory.

168. We say, also, when the relation is *direct,* that one quantity of each species is in *direct proportion* with its correspondent; and if the relation is inverse, that each quantity is in inverse proportion with its correspondent. Thus, for example,

Two fractions of the same denominator are in *direct* proportion with their numerators.

For we have seen that if the numerator is rendered double, triple, quadruple, ... or one half, one quarter, one third of what it is, the fraction will be rendered two, three, four . . . . times greater or less than it was.

By an analogous process, we could prove, that two fractions, having the same numerators, are in the *inverse proportion* of their denominators.

When the fractions have different numerators and denominators, we commence by reducing them to the same denominator or to the same numerator, and the question is thus reduced to one of the two preceding cases.

We are then led to a new mode of expression, which consists in saying that the given fractions are in *Compound Proportion*, direct or inverse, of the two products of the *numerator* of the *first* by the *denominator* of the *second*, and of the *numerator* of the second by the *denominator* of the first.

In order to justify this mode of expression, let us consider, for example, the two fractions, $\frac{3}{7}$ and $\frac{4}{11}$.

Reducing them to the same denominator, we obtain

$$\frac{3 \times 11}{7 \times 11} \text{ and } \frac{4 \times 7}{7 \times 11};$$

and these two fractions are in the direct ratio of 3×11 to 4×7, or of 33 to 28. If, on the contrary, we reduce them to the same numerator, they become

$$\frac{3 \times 4}{7 \times 4} \text{ and } \frac{3 \times 4}{3 \times 11};$$

and in this case the two fractions are in the inverse ratio of the denominators, or the first is to the second as 3×11 is to 7×4, or as 33 is to 28, the same as before.

But we see that the two terms of this ratio are the one, the product of the *numerator* of the *first* fraction by the *denominator* of the *second*; the other, the product of the *numerator* of the *second* by the *denominator* of the *first*.

This compound ratio is in some sort the result of the multiplication of two simple ratios, which are either direct or inverse with regard to each other.

169. *Applications.* — As application of what has just been said, we will indicate the method of bringing into a proportion certain *surfaces* and *volumes* or *solids*, because there is a number of questions in which we have need of these numerical valuations.

Let it be required to compare the *superficial extent* of two pieces of stuff, one of which is 24 yards *long* by $\frac{2}{3}$ yard *wide;* the other, 17 yards *long* by $\frac{5}{4}$ *wide.*

By a process of reasoning analogous to that used in (166), we see that 24 yards *long* by $\frac{2}{3}$ yards *wide,* is the same thing as $24 \times \frac{2}{3}$ yards *long* by 1 yard *wide.*

In the same manner, 17 yards long by $\frac{5}{4}$ of a yard wide, is equal to $17 \times \frac{5}{4}$ long by 1 yard wide.

Then, since the breadth of the two pieces is the same, the *ratio* of the two superficial extents is equal to that of the two lengths, and we find this ratio to be

$$24 \times \tfrac{2}{3} : 17 \times \tfrac{5}{4}, \text{ or } \frac{24 \times 2}{3} : \frac{17 \times 5}{4}, \text{ or } \frac{24 \times 2 \times 4}{3 \times 4} : \frac{3 \times 17 \times 5}{3 \times 4};$$

or, simplifying, as 64 : 85.

Again, let there be two rolls of paper-hangings, one of which is 15 yards long, by $\frac{4}{5}$ of a yard wide; the other 19 yards long, by $\frac{7}{8}$ of a yard wide; we would find in the same manner for the ratio of the superficial extents of the two rolls,

$$15 \times \tfrac{4}{5} : 19 \times \tfrac{7}{8}, \text{ or } \frac{15 \times 4}{5} : \frac{19 \times 7}{8}, \text{ or } \frac{15 \times 4 \times 8}{5 \times 8} : \frac{19 \times 7 \times 5}{5 \times 8}; \text{ or,}$$

simplifying, 96 : 133.

We conclude from this, that, whenever the enunciation of a question gives rise to a comparison of superficial extents, *in order to reduce them to the unit of length, we must form the product of the length by the breadth, and then compare the resulting quantities.*

As to volumes or solids, it will suffice to take one example, in order to determine the steps to be followed.

Required to determine *the ratio in cubic yards of the solid contents of two pieces of masonry?*

We suppose that the first piece is 60 yards long, by $\frac{3}{4}$ of a yard thick, and 3 yards high; and the second, 125 yards long, by $\frac{7}{8}$ of a yard thick, and $4\frac{1}{2}$ yards high.

Reasoning, as in the preceding case, we find that for the first wall it is as if it was $60 \times \frac{3}{4} \times 3$ yards long, by 1 yard thick, and 1 yard high; and for the second, $125 \times \frac{7}{8} \times \frac{9}{2}$ yards long, by 1 yard thick, and 1 yard high. In other words, the two walls must contain, respectively, $60 \times \frac{3}{4} \times 3$ cubic yards, and $125 \times \frac{7}{8} \times \frac{9}{2}$ cubic yards. Then, the ratio of the two volumes is equal to that of these two products, or of

$$\frac{60 \times 3 \times 3 \times 4}{16} \text{ to } \frac{125 \times 7 \times 9}{16}, \text{ or of } 48 \text{ to } 175.$$

Whence we see, that in order to obtain the two pieces of work expressed in cubic yards, it suffices to form for each one of them the product of the length by the thickness and by the height. After which we easily find the *ratio* of the two.

Compound Rule of Three.—General Method of Reduction to Unity.

170. The enunciation of a question often contains more than four numbers, between which it becomes necessary to establish either direct or inverse proportions; and thus arise the distinctions, *Single Rule of Three*, and *Compound* or *Double Rule of Three.* These names arise from the mode of resolution, which is an application of the theory of proportions. But this mode has been generally replaced by the method called the method of *Reduction to Unity*, which we will now develop, remarking that the second mode of resolution of the problems in (166) is a particular case of this general method.

171. *Third Problem.—It requires* 1800 *yards of cloth,* $\frac{5}{4}$ *of a yard wide, to clothe* 500 *men. Required the number of yards of cloth,* $\frac{7}{8}$ *of a yard wide, which shall clothe* 960 *men?*

Table of Calculation.

1800	yards long,	$\frac{5}{4}$	wide,	500	men.
x	"	$\frac{7}{8}$	"	960	"
$1800 \times \frac{5}{4}$	"	1	"	500	"
$x \times \frac{7}{8}$	"	1	"	960	"
$\frac{1800 \times 5}{4 \times 500}$	"	1	"	1	"
$\frac{x \times 7}{960 \times 8}$	"	1	"	1	"

$$\text{Then, } \frac{x \times 7}{960 \times 8} = \frac{1800 \times 5}{4 \times 500}.$$

Analysis. — After arranging upon two horizontal lines the six numbers which the enunciation contains, and of which the number of yards required forms part, we reason in the following manner: 1800 yards long, by $\frac{5}{4}$ wide, and x yards long, by $\frac{7}{8}$ wide, are the same thing as $\frac{1800 \times 5}{4}$, and $\frac{x \times 7}{8}$ yards long, by 1 yard wide.

We write, then, these numbers upon two new lines, preserving the numbers 960 and 500 in their respective places in the two new lines. Since, with $\frac{1800 \times 5}{4}$ yards long, by 1 yard wide, we can clothe 500 men, one man could be clothed with $\frac{1800 \times 5}{4 \times 500}$. In the same manner, if $\frac{x \times 7}{8}$ yards can clothe 960 men, one man could be clothed with $\frac{x \times 7}{8 \times 960}$, which gives again two new lines, which we place below the preceding. Now, the two last expressions which we have just obtained, representing *both* the quantity of cloth necessary to clothe one man, are necessarily equal. We have then

$$\frac{x \times 7}{8 \times 960} = \frac{1800 \times 5}{4 \times 500};$$

or, reducing to the same denominator, and then suppressing this denominator,

$$x \times 7 \times 4 \times 500 = 1800 \times 5 \times 8 \times 960.$$

Dividing the two members of the equation by the multiplier of x, we have

$$x = \frac{1800 \times 5 \times 8 \times 960}{7 \times 4 \times 500} = \frac{36 \times 960}{7} = \frac{34560}{7} = 4937\tfrac{1}{7};$$

that is, it would require $4937\frac{1}{7}$ yards to clothe 960 men.

Verification.

$\frac{1800 \times 5}{4 \times 500}$ reduces, obviously, to $\frac{18}{4}$, or $4\frac{1}{2}$;

on the other hand,

$$\frac{34560}{7} \times \frac{7}{8 \times 960},$$

reduces also to $4\frac{1}{2}$. The number $4\frac{1}{2}$, or 4 yards and a half, expresses in the two cases the quantity of cloth necessary to clothe one man.

172. *Problem Fourth.* — 500 *men, working* 12 *hours a day, employed* 57 *days in excavating a canal* 1800 *yards long, by* 7 *yards wide, by* 3 *yards deep; required in how many days* 860 *men, working* 10 *hours a day, can dig another canal* 2900 *yards long, by* 12 *wide, and* 5 *deep, in an earth* 3 *times as difficult to excavate as the first.* (This is one of the most complicated questions which can be given in this Compound Proportion, or Rule of Three.)

Table of Calculations.

500 men.	12 hours.	57 days.	$(1800 \times 7 \times 3 \times 1)$	cubic yards.
860 "	10 "	x "	$(2900 \times 12 \times 5 \times 3)$	"
1 man	1 hour	1 day	$\left\{\frac{1800 \times 7 \times 3 \times 1}{500 \times 12 \times 57}\right\}$	"
1 "	1 "	x "	$\left\{\frac{2900 \times 12 \times 5 \times 3}{860 \times 10}\right\}$	"

$$\text{Then, } x=\frac{2900\times 12\times 5\times 3}{860\times 10}, \text{ divided by } \frac{1800\times 7\times 3\times 1}{500\times 12\times 57}; \ (1)$$

$$\text{or, } x=\frac{2900\times 12\times 5\times 3\times 500\times 12\times 57}{860\times 10\times 1800\times 7\times 3\times 1}.$$

Analysis. — It is necessary, first, according to what has been laid down in (169), to convert into cubic yards the two pieces of work; the one already executed, and the other to be performed. This we do by multiplying together the length, breadth, and depth in each case. Besides, since, according to the enunciation, the earth of the second is three times more difficult to excavate than the first, if we express by 1 and 3 the relative difficulties, we must introduce into the two products, of which we have just spoken, the factors 1 and 3.

This established, after having placed, as in the preceding problem, all the numbers comprised in the enunciation upon two different lines, we are led, by a course of reasoning entirely similar to that which we pursued in the solution of the third problem, to form two new lines representing,—*the one*, the work done by 1 man in one hour and in one day; *the other*, the work done by 1 man in one hour and in x days.

Now, it is clear, that these two quantities of work must bear to each other the direct proportion of the two periods employed to perform them. We have then the equality (1) given in the table of the calculations, whence we deduce the final equation there given; and, effecting all the operations indicated, this equation gives, finally,

$$x=549\tfrac{51}{301};$$

that is to say, it would require 549 days, and $\frac{51}{301}$, or about $\frac{1}{6}$ of a day, for 860 men to excavate the second canal.

173. The problems which precede, suffice to exhibit the steps to be followed when the method of *Reduction to Unity* is employed.

But it may be useful, perhaps, to consider the results furnished by the last two problems, in order to deduce from them some new consequences concerning the use of direct and inverse ratios.

The analysis of the problem in (171) led to an expression for the number of cubic yards sought,

$$x = \frac{1800 \times 5 \times 8 \times 960}{7 \times 4 \times 500}.$$

Now, if we go back to the enunciation of the question, in order to distinguish the *correspondents* of each species, and if we separate by means of the sign of multiplication (×) the different ratios of each term and its *correspondent*, we shall be able to place the preceding expression under the form

$$\frac{x}{1800} = \tfrac{5}{4} \times \tfrac{8}{7} \times \frac{960}{500}.$$

or again, under this,

$$\frac{x}{1800} = \frac{\frac{5}{4}}{\frac{7}{8}} \times \frac{960}{500}.$$

Examining the product in the second member, we see that the second factor, which is the ratio of the two numbers of men to be clothed, is *direct* with that of the numbers of yards of cloth, $\frac{x}{1800}$; while the first factor, or the ratio of the two breadths, is *inverse* with the same ratio, $\frac{x}{1800}$; thus, this last *ratio*, called compound (168), is equal to the product of the ratios of the two numbers of men, and of the two breadths, direct for the *men*, and inverse for the *breadths*. And, in fact, the more men there are to clothe, the more cloth necessary; but, the wider the cloth, the smaller number of yards necessary to make a given quantity.

The expression obtained in the problem of (172),

$$x = \frac{2900 \times 12 \times 5 \times 3 \times 500 \times 12 \times 57}{860 \times 10 \times 1800 \times 7 \times 3 \times 1},$$

can be put under the form

$$\frac{x}{57} = \frac{2900}{1800} \times \tfrac{12}{7} \times \tfrac{5}{3} \times \tfrac{3}{1} \times \frac{500}{860} \times \frac{12}{10};$$

and we see also, in this case, that the ratio of the two numbers of days necessary for the performance of the two pieces of work is equal to the product of the ratios of the correspondents of each species; *direct* in the case of the dimensions of the canals and the difficulties of the excavation; but inverse for the numbers of workmen employed, and the numbers of hours per diem which they laboured.

Whence we can give this sort of General Rule for the resolution of every question whose enunciation contains proportional quantities:

Form a product of all the ratios, direct or inverse, of the correspondents of each species, excepting the ratio of which the quantity sought forms one part; then equal this product to the ratio of the quantity sought to the quantity of the same species with itself.

We obtain thus the expression of the equality of two ratios, from which we easily deduce the value of the unknown.

Rule of Simple Interest.

174. *The Simple Interest on a sum of money is the profit arising from the loan of this sum for a certain time.*

The sum lent, or placed out *at interest*, is called the *Principal* or *Capital.*

The *interest* upon a sum of money depends upon the amount of the Principal, upon the time for which it is lent, and upon what is called the *rate of interest*, or the *interest* which a certain fixed sum bears for a given fixed period.

Ordinarily, the *rate* is, in the United States, the interest which the sum of one hundred dollars bears in one year, and hence is called the rate *per cent.*

This *rate*, which we consider a sort of unit of interest, is purely conventional, and depends generally on the abundance or scarcity of capital. Nevertheless, there are, in commerce and banking, certain limits (in most countries fixed by law), beyond which the rate becomes usury.

It is evident that the interest on two principals for the same period must be proportional to the principals, (the rate being constant), and the interest on the same principal for two different periods, are proportional to the lengths of the periods.

Whence it follows, that the rule of interest is only a particular case of the *Rule of Three.*

Thus, the questions which arise under it can be treated in the same manner as the preceding.

175. *Example.* — *Required, the Interest on* \$4500 *for* 2 *years and* 5 *months, at the rate of* \$7 *for every* \$100; *or, by abbreviation, at the rate of* 7 *per cent. per annum.*

This enunciation can be thus rendered: \$100 bring \$7 in one year, or 12 months; how much ought \$4500 to bring in 2 years and 5 months, or 29 months?

The numbers can be thus arranged:

$$\begin{array}{rll} 100 & 12 \text{ months} & 7 \\ 4500 & 29 \quad \text{``} & x \\ 1 & 1 \text{ month} & \dfrac{7}{100\times12} \\ 1 & 1 \quad \text{``} & \dfrac{x}{4500\times29}. \end{array}$$

The quantities,

$$\frac{7}{100\times12} \text{ and } \frac{x}{4500\times29},$$

express each what one dollar brings in one month, and must therefore be equal, and we have,

$$\frac{x}{4500\times29}=\frac{7}{100\times12};$$

whence,

$$x=\frac{4500\times29\times7}{100\times12}=\frac{15\times29\times7}{4}.$$

Reducing to decimals,

$$x=\$761{\cdot}25,$$

the interest on $4500 for 2 years and 5 months, at 7 per cent. per annum.

176. Generally, let us denote the principal by a, the time by t, the *rate per cent.* per annum by i and by g, the interest on the capital, by a. We shall have,

$100	1 year	1 dollar.	
a	t	g	
1	1	$\frac{i}{100}$	interest on $1 for 1 year.
1	1	$\frac{g}{a\times t}$	“ “

Then,
$$\frac{g}{a\times t}=\frac{i}{100},$$
and, consequently,
$$g=\frac{a\times i}{100}\times t. \quad (1)$$

The time t can be a fractional number of the unit, *year* having for denominator the number of *months* or of *days* in the year. If we place (1) under the form

$$g=\frac{a\times i}{100}\times t,$$

it can be translated into the following rule:

In order to determine the interest g, *multiply the given principal by the rate of interest for one year, and divide the product by* 100; *then multiply the result by the number of years, fractional or entire.*

Example.—Required, the interest on $2524·65, *at* $4\frac{1}{2}$ *per cent. per annum for* 2 *years and* 7 *months.*

We have, first,

$$2524{\cdot}65\times 4{\cdot}5=11360{\cdot}925.$$

Dividing by 100, 113·60925

	2 yrs. 7 mos.
For two years,	227·21850
" 6 months,	56·804625
" 1 month,	9.467437
	293·490562 ; or, $293·49.

It is obvious that this division by 100 can be performed on the *rate* before the first multiplication, thus converting that into a decimal fraction, by which the principal is to be multiplied.

Example.—Required, the interest on $365·874, *at* 5½ *per cent. for one year?*

This rate, 5½ per cent., divided by 100, gives 0·055. We then multiply 365·874 by 0·055.

$$\begin{array}{r} 365{\cdot}874\\ {\cdot}055\\ \hline 1829370\\ 1829370\\ \hline \$20{\cdot}12307 \end{array}\qquad \$20{\cdot}12.\ \textit{Ans.}$$

177. This second method, which we have applied in the last two examples, is always to be preferred, especially when we wish to determine the interest for a certain number of days.

Required, for example, to find the interest on $1748·19, for 113 days, at 4¾ per cent. per annum. (We suppose the year to contain 360 days, 30 days for each month).

We multiply 1748·19 by 4¾, divide by 100; we then divide 113 into 60+30+20+3 days, and find the interest for each one of these parts separately. Summing these parts, we have the interest required.

Table of Calculations.

	1748·19	
	$4\frac{3}{4}$	
	6992·76	
	$\frac{1}{2}$ 874·095	
	$\frac{1}{4}$ 437·0475	
	8303·9025	
and dividing by 100,	83·039025	for one year's interest.
For 60 days,	13·839837	
" 30 "	6·919918	half of the above.
" 20 "	4·613279	$\frac{1}{3}$ " "
" 3 "	0·691992	$\frac{1}{10}$ of the int. for 30.
	26·065026	

Thus, the interest on $1748·19 for 113 days, is $26·06.*

178. The equation (1) of (176), contains the solutions of four different problems.

1st. *Knowing the Principal, time and rate, to find the Interest.*

This we have discussed in several examples.

2d. Knowing the Interest, time, and rate, to find the Principal.

3d. Knowing the Interest, Principal, and time, to find the rate.

4th. Knowing the Principal, Interest, and rate, to find the time.

All these admit readily of solution; but we will limit ourselves here to an example of the fourth problem, treating it by both of the methods explained in a preceding article.

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* The rate of 6 per cent. per annum admits of the following abbreviation of the above rules when applied to a given number of months; 6 per cent. per annum is $\frac{1}{2}$ per cent. per month, or 1 per cent. for two months. Then we can say, in order to find the interest on a certain principal for a given number of months, at the rate of 6 per cent. per annum, we *multiply the principal by $\frac{1}{2}$ the number of months, and divide by* 100.

A sum of \$2524·65 brought \$293·49, at the rate of $4\frac{1}{2}$ per cent. per annum. Required the length of time the sum was placed at interest?

First Mode of Proceeding.

$$\begin{array}{ccc}
100 & 1 \text{ year} & 4\frac{1}{2} \\
2524{\cdot}65 & t & 293{\cdot}49 \\
\hline
1 & 1 & \frac{4{\cdot}50}{100} \\
1 & t & \frac{293{\cdot}49}{2524{\cdot}65}
\end{array}$$

$$\frac{t}{1} = \frac{293{\cdot}49}{2524{\cdot}65} \times \frac{100}{4{\cdot}5} = \frac{29349}{252{\cdot}465 \times 45} = \frac{29349000}{11360925}.$$

Effecting the division, we obtain 2 years and 7 months, neglecting a fraction less than 0·001 of a month.

Second Method.

$$\begin{array}{lr}
 & 2524{\cdot}65 \\
 & 4\frac{1}{2} \\
\cline{2-2}
 & 10098{\cdot}60 \\
 & 1262{\cdot}325 \\
\cline{2-2}
 & 11360{\cdot}925 \\
\text{or, dividing by 100,} & 113{\cdot}60925 \quad \text{interest for one year.}
\end{array}$$

And as \$293·49 is the interest for t years, we must divide 29349000 by 11360925, in order to obtain the time required, t.

Rule of Discount.

179. *Discount is the deduction which is made from an amount payable at the end of a certain time, when we wish to make it payable at the present time, or before it falls due by agreement.* It is usually, in bankers' terms, the deduction which we make from the *face* (amount of a promissory note, in order to get its cash value. This reduction is usually made at so much in the

hundred per annum; and this is the *rate per cent.* of discount. The discounter is he who cashes the note by anticipation.

It is easy to see that the rule of discount is the same with the rule of interest, with this difference, that, in the latter case, the *borrower* is obliged to restore to the *lender* the sum lent, increased by its interest; while, in the case of discount, the possessor or maker of the note receives only the difference between the amount of the note and the discount which is made by reason of the anticipation of its payment.

Example First. — *Required, the discount on a note of* $875·49, *payable in* 18 *months, at the rate of* 4·8 *per cent. per annum.*

First Method.

$$\begin{array}{lllll}
\$100 & 12 \text{ months} & 4{\cdot}80 & & \\
875{\cdot}49 & 18 \quad `` & x & & \\
\hline
1 & 1 \quad `` & \dfrac{4{\cdot}80}{1200} & \text{discount on \$1 for 1 year.} & \\
1 & 1 \quad `` & \dfrac{x}{875{\cdot}49\times18} & `` \qquad `` &
\end{array}$$

Then, $$\frac{x}{875{\cdot}49\times 18}=\frac{4{\cdot}80}{1200};$$

whence, $$x=\frac{4{\cdot}80\times 875{\cdot}49\times 18}{1200}=\frac{40\times 87549\times 18}{1000000};$$

or, performing the calculations,

$$x=63{\cdot}035280=63{\cdot}04.$$

Amount of the note, $875·49.
Discount, ... 63·04.
Difference, . . . $812·45, the amount which the discounter pays.

Second Method.

Amount of note,	$875·49	
Rate of discount per an.	4·8	
	700392	
	350196	
	4202·352	
dividing by 100,	42·02352	
1 year, 6 months.		
1 year,	42·02352	
6 months,	21·01176	
	63·03528	as above.

This example suffices to show the identity of the calculations under the Rules of Interest and Discount.

Example Second. — Required, the discount on a note of $3478·19, *payable in* 286 *days, the rate being* 6·25 *per cent. for* 360 *days.*

We commence by decomposing the number 286 into its parts,

$$180 + 90 + 10 + 5 + 1.$$

We then make the following table of calculations:

3478·19			
6·25			
1739095			
695638			
2086914			
217·386875	discount for	360	days.
108·693437	"	180	"
54·346719	"	90	"
6·038524	"	10	"
3·019262	"	5	"
0·603852	"	1	"
172·701794	discount for	286	days.
$3478·19			
172·70			
$3306·49	cash value.		

180. The generalization of the rule of discount would lead to the equation

$$(2) \quad \mathrm{E} = \frac{a \times e \times t}{100},$$

in which e E would designate the discounts on \$100, and on the amount of the note respectively. These letters would simply replace g and i of (176).

We could, according to the equation (2), establish the enunciations of four general problems analogous to those of (178).

181. There is another rule of discount which we cannot pass by; for although it is not generally employed, it appears more rational and more just.

One example will suffice to give an idea of this second mode of discounting.

A note of \$1500, *payable at the end of* 15 *months, is presented to a banker, who agrees to cash it at a discount of* 4·6 *per cent. per annum. Required, what the holder of the note must receive?*

Analysis. — Admit, that 4·60, the rate of discount, is at the same time the rate of interest of a sum put out at interest.

It is clear that the possessor of the note ought to receive now a sum which, placed at interest at the rate of 4·6 per cent. per annum for 15 months, would give him, capital and interest added, the amount of his note.

Now, the interest of \$100 for one year, being 4·60, becomes, for 15 months, 4·60 + $\frac{1}{4}$ of \$4·60, or \$5·75.

This proves that \$100, placed out at interest, would, at the end of 15 months, become \$105·75, capital and interest.

Consequently, \$105·75, payable in 15 months, are equivalent to \$100 payable now; then \$1, payable in 15 months, is equal to $\frac{100}{105{\cdot}75}$, payable now; and, consequently, \$1500, payable in 15 months, can be represented by

$$\frac{100 \times 1500}{105{\cdot}75}, \text{ or } \frac{15000000}{10575}, \text{ or } \$1418{\cdot}43{\cdot}97,$$

payable now.

Whence it follows, that the holder of the note ought to receive from the banker a sum of $1418·44.

In fact, if we calculate by the Rule of Interest, what $1418·44 ought to bring at the end of 15 months, at the rate of 4·60 per cent. per annum, we obtain

$$g = 81{\cdot}5603,$$

which, added to $\underline{1418{\cdot}4397,}$

gives $1500·0000, the amount of the note.

Now, instead of following this method, the banker determines the interest on $1500 for 15 months, at 4·6 per cent., which gives

$86·25;

and this he subtracts from $\underline{\$1500{\cdot}00,}$

$1413·75, the difference which he gives the possessor of the note.

N. B. It is to be remarked, that the excess of $86·25 over $81·56, or $4·69, which the banker gains by the last operation, is nothing more than the interest on $81·56 for 15 months. For, multiplying $81·56 by 5·75, (rate for 15 months,) and dividing by 100, we obtain $4·6897, or $4·69.

This advantage which the banker gains, independently of the profit which belongs to him of right, is a sheer loss on the part of the holder of the note.

There is a way of operating, according to the first rule, without injury to the interests of the possessors of notes. This would be to establish a rate of discount a little lower than the legal rate of interest; but the difficulty would be to proportion the one to the other fairly under all circumstances.

We give the two rules or enunciations of the two methods which we have given above.

1st. (179). *Calculate the interest on the amount named in the note, from the present time to the date at which it falls due; then* *subtract this interest from the amount named in the note. This will be the cash value of the note.*

2d. (181). *Find what* $100, *placed out at interest for the given time will bring, capital and interest added; then multiply the amount named in the note by the ratio of* $100 *to this sum; the quotient will be the present value of the note.*

The first rule is generally received in commerce, because it is more expeditious and convenient with regard to the calculations. It is, moreover, a matter of agreement between the banker and holder of the note.

The Questions of Compound Interest and Discount, and the subject of Annuities, require a knowledge of the use of Logarithms, in order to be thoroughly discussed. Hence, we pass them by here, merely adding, that, in Compound Interest, the interest is added to the principal at the end of the year, or period chosen as unit; and then this sum is regarded as a new principal, on which the interest is calculated for the given period, and again added, &c., &c.

There are a great number of questions, such as Insurances, Rents, &c., &c., which come under the rule of per centage, but they present no difficulty to the student who understands thoroughly the preceding discussions of proportional quantities. They are generally given in full in the Commercial Arithmetics.

Rule of Fellowship.

182. The Rule of Fellowship has for its object,

To divide among several persons associated in a partnership business the profit or loss which results from their enterprise.

It is generally admitted, (and it is moreover conformable to equity,) that the part of gain or loss of each partner is — 1st, proportional to the amount of capital he has placed in the business, when the times are equal; 2d, proportional to the time when the amounts invested are the same.

From this it results that, for different capitals and different times, the parts are proportional to the products of the capital stocks by the times; since, by multiplying the stocks by the times respectively, we bring them back to amounts invested for the same time. Thus, the question, considered under the most general point of view, is, *to divide a given number into parts directly proportional to other numbers also given.*

Problem First. — *Three persons are associated in trade. The first puts* \$15,000 *in the common stock; the second,* \$22,540; *and the third,* \$25,600. *At the end of one year, the profits of the enterprise are* \$12,000. *Required, the share of each one of the partners?*

Analysis. — The sum of the three amounts invested in trade being \$63,140, we reason in the following manner:

\$63,140 have given a profit of \$12,000; then \$1 has produced $\frac{12000}{63140}$ dollars profit. Then, for

$$15000\ .\ .\ .\ .\ \text{we have}\ \frac{12000}{63140}\times 15000=\frac{18000000}{6314}=2850{\cdot}807.$$

$$22540\ .\ .\ .\ .\qquad \text{``}\qquad \frac{12000}{63140}\times 22540=\frac{27048000}{6314}=4283.813.$$

$$25600\ .\ .\ .\ .\qquad \text{``}\qquad \frac{12000}{63140}\times 25600=\frac{30720000}{6314}=\underline{4865{\cdot}378.}$$

$$11999{\cdot}998.$$

Thus, the first person must receive \$2850·81; the second, \$4383·81; and the third, \$4865·38.

And these three sums, added, reproduce the total gain, \$12,000.

Problem Second. — *A capitalist commences an enterprise with a stock of* \$25,000. *Five months later, a second capitalist joins the enterprise, and furnishes an additional capital of* \$40,000. *Six months after this first addition, a third capitalist adds* \$60,000. *At the end of two years the partnership is dissolved,* *after having realised a profit of* \$76,000. *Required, the share of each partner?*

The \$76,000 are to be divided among the partners proportionally to the products of their respective investments, by the numbers of months during which these funds were in the enterprise.

Now, 1st, \$25,000, invested for 24 months, equal 25000 × 24, or \$600,000 vested for 1 month; 2d, \$40,000 invested for 19 months, are equivalent to \$760,000 for 1 month; 3d, \$60,000 for 13 months, are equivalent to \$780,000 invested for 1 month. The question is then the same as the first. Having formed the sum of the three amounts invested = \$2140000, we obtain successively for the three parts or shares of the profit,

$$\text{First share,}\quad \frac{76000}{2140000}\times 600000 = 21308{\cdot}411.$$

$$\text{Second share,}\quad \frac{76000}{2140000}\times 760000 = 26990{\cdot}654.$$

$$\text{Third share,}\quad \frac{76000}{2140000}\times 780000 = \underline{27700{\cdot}934.}$$

$$75999{\cdot}999.$$

The shares are, respectively,

\$21308·42; \$26990·65; \$27700·93.

183. In general, let it be required to divide any number, a, into parts proportional to the given numbers, $m, n, p, q \ldots .$

Form, first, the sum of the numbers, $m, n, p, q \ldots .$ then, multiply each one of these numbers by the ratio

$$\frac{a}{m+n+p+q+\ldots .}$$

We obtain, thus,

$$\frac{a\times m}{m+n+p+\ldots .},\quad \frac{a\times n}{m+n+p+\ldots .},\quad \frac{a\times p}{m+n+p+\ldots .},$$

fractions, which have the same denominator, and are necessarily in the direct proportion of their numerators, or because of the common factor, a, in the direct proportion of $m, n, p, q \ldots .$

When the numbers, m, n, p, q . . . . are fractional, we commence by reducing them to the same denominator, and then the question becomes the same as the preceding.

Divide 360 into four parts, proportional to the numbers

$$\tfrac{2}{3},\ \tfrac{7}{8},\ \tfrac{11}{12},\ \tfrac{17}{32}.$$

These fractions, reduced to the least common denominator, become

$$\tfrac{64}{96},\ \tfrac{84}{96},\ \tfrac{88}{96},\ \tfrac{51}{96}.$$

Then, the four parts must be respectively proportional to the numbers 64, 84, 88, 51.

The sum of these numbers being 287, we have, successively,

$$\begin{array}{lllr}
\text{For the first part,} & & \tfrac{360}{287}\times 64 = & 80{\cdot}28. \\
\text{"} & \text{second,} & \tfrac{360}{287}\times 84 = & 105{\cdot}37. \\
\text{"} & \text{third,} & \tfrac{360}{287}\times 88 = & 110{\cdot}38. \\
\text{"} & \text{fourth,} & \tfrac{360}{287}\times 51 = & 63{\cdot}97. \\
\hline
 & & & 360{\cdot}00.
\end{array}$$

184. The following questions belong also to the same rule:

Problem Third.—Required, to divide a sum of \$36,000 *among four persons, so that the second shall have twice as much as the first; the third as much as the first two together; the fourth three times as much as the third.*

We can make the first share a principal unit, with which we compare the rest. Calling, then, the first part 1, the second part will be 2, the third 3, and the fourth 9, by the conditions of the question.

The question is then to divide \$36,000 into four parts, proportional to the numbers 1, 2, 3, 9. We obtain for the four parts,

$$\begin{array}{llll}
\text{First part,} & \dfrac{36000}{1+2+3+9}\times 1 & \text{or } \dfrac{36000}{15} & = 2400 \\[2ex]
\text{Second part,} & \dfrac{36000}{15}\times 2 & \text{"} & = 4800 \\[2ex]
\text{Third part,} & \dfrac{36000}{15}\times 3 & \text{"} & = 7200 \\[2ex]
\text{Fourth part,} & \dfrac{36000}{15}\times 9 & \text{"} & = 21600
\end{array}$$

Problem Fourth. — *A person leaves* \$40,000, *to be divided among four heirs, so that the first shall have* $\frac{1}{6}$ *of the whole; the second* $\frac{2}{5}$; *the third* $\frac{4}{9}$; *the fourth* $\frac{1}{3}$. *Required, the share of each heir.*

If the sum of the four fractions was exactly equal to 1, the conditions of the bequest would be fulfilled by taking successively $\frac{1}{6}$, $\frac{2}{5}$, $\frac{4}{9}$, and $\frac{1}{3}$, of \$40,000. But, if we reduce these fractions to the same denominator, we find

$$\frac{15}{90},\ \frac{36}{90},\ \frac{40}{90},\ \frac{30}{90},$$

the sum of which is greater than 1. Hence, the bequest would be more than absorbed by the three first parts. But if the \$40,000 is to be divided proportionally to the four numbers, $\frac{1}{6}$, $\frac{1}{2}$, $\frac{4}{9}$, $\frac{1}{3}$, we would simply have to divide it into parts proportional to the numbers 15, 36, 40, and 30, the same as the problem in (183).

185. We add here a rule which has for its object to determine the relative value of the coins of two countries, knowing the proportions between these coins and those of other countries. It consists in reducing to a single proportion, by multiplication, several given proportions. It is really nothing more than an application of the rule of compound fractions, or fractions of fractions.

A single example will suffice to give an idea of the rule and the mode of applying it.

EXAMPLE.

48 francs . . .	are equal to	39 English shillings.
13 English shillings	"	8 German florins.
50 German florins	"	9 ducats of Hamburg.
15 ducats of Hamburg	"	43 roubles, Russian.

How many Russian roubles are equal in value to 2500 francs?

If 48 francs are worth 39 shillings, then 1 franc is worth $\frac{39}{48}$ of a shilling. In the same manner, if 13 shillings are worth 8 florins, 1 shilling is worth $\frac{8}{13}$ of a florin; and, consequently, 1 franc is worth $\frac{39}{48}$ of $\frac{8}{13}$ of a florin. Again, if 50 florins are worth 9 ducats, then 1 florin equals $\frac{9}{50}$ of a ducat. Continuing this reasoning, we find that

$$2500 \text{ francs} = 2500 \text{ times } \tfrac{39}{48} \text{ of } \tfrac{8}{13} \text{ of } \tfrac{9}{50} \text{ of } \tfrac{43}{15} \text{ of a rouble.}$$

$$\text{Then, } 2500 \text{ francs} = \frac{39 \times 8 \times 9 \times 43 \times 2500}{48 \times 13 \times 50 \times 15} \text{ roubles.}$$

Rule of Alligation.

186. The questions which come under this rule are of two sorts:

We may *either wish to find the mean value of several sorts of things, knowing the number and particular value of each sort, or it may be required to determine the quantities of several sorts of things which must enter into a mixture, knowing the price or value of each sort, and the price or total value of the mixture.*

We will discuss only the questions of the first nature; the second belonging to the province of algebra.

Example First. — *A wine merchant has mixed wines of different qualities, viz.,* 250 *pints, at* 60 *cents the pint;* 180 *pints, at* 75 *cents; and* 200, *at* 80 *cents. Required, the price of one pint of the mixture?*

We observe, first, that

250	pints,	at 60	cents,	bring	$150
180	"	at 75	"	"	$135
200	"	at 80	"	"	$160
					$445

Giving $445 for the total price of the three quantities of wine mixed.

If, now, we form the sum 630 of the three numbers, 250, 180, and 200, the question will obviously be reduced to the following:

$$\begin{array}{r} 250 \\ 180 \\ 200 \\ \hline 630 \end{array}$$

630 pints of wine cost $445; what is the cost of each pint?

71 cents is the price required.

General Rule. — *In order to find the price of the principal unit of a mixture* — 1st. *Multiply the price of this principal unit of each sort of thing by the number of units of this sort, and add all the products.* 2d. *Sum up the numbers of units of these different sorts.* 3d. *Divide the sum of the products or the total price by the sum of the numbers of units.*

Or, more briefly — *Find the total price of the mixture by summing up the prices of its parts. Then divide this total price by the number of principal units in the mixture. We thus obtain the price of one principal unit.*

Example Second. — We wish to melt together 23 kilogrammes of silver, 826 thousandths fine; 14 kilogrammes 910 thousandths fine; and 19 kilogrammes 845 thousandths fine. Required, how many thousandths fine the mixture will be? That is, how many parts of pure silver each 1000 parts of the new coin will contain? (We say an ingot of gold or silver is $\frac{9}{10}$, or 880 thousandths, &c., fine, when $\frac{9}{10}$, or 880 thousandths of it is pure silver or gold.) It results, then, from the enunciation, that

1st.	23 *k.*	at ·825 = 23 × ·825, or	18·975 *k.*	of pure silver.
2d.	14 *k.*	at ·910 = 14 × ·910, or	12·740 *k.*	"
3d.	19 *k.*	at ·845 = 19 × ·845, or	16·055 *k.*	"
	56		47·770	"

Then, the 56 kilogrammes of the mixture contain 47·770 kilogrammes of pure silver. Thus, the fineness of the new ingot will be expressed by $\frac{47{\cdot}770}{56}$, or 0·853; that is, it is 853 thousandths fine.

187. *Mean or average values.* — The determination of the *mean values* of several things of different values, is a particular case of the rule of alligation of the first sort.

We call the mean value of several things whose particular values are already known, *the sum of the values of these things divided by their number.* Thus, in the case of two things, the mean value is the half sum of the values of these things.

Example Third.—The length of a park was measured four different times. The first measurement gave 250·439 metres; the second, 250·695 metres; the third, 249·750 metres; finally, the fourth, 251·158 metres. Required, the length of the park?

As none of the measurements agree, it is clear that the only means of answering the question is to find the average or mean value of all these measurements. We find for their sum, 1002·042; dividing this result by 4, we obtain 250·5105 metres for the mean.

Problems which, without depending on fixed or General Rules, can nevertheless be resolved arithmetically.

188. In the preceding questions, the methods of arriving at the required solution are fixed and general; that is to say, susceptible of being applied to all questions of the same nature. But an infinite number can be proposed which come only in part under these methods, or do not in any manner depend upon them. In these cases, algebra alone furnishes sure and direct methods of resolution. Nevertheless, we will show how these sorts of questions can be resolved arithmetically. We have seen, (154), that, in order to analyse or resolve a problem, we must, *by reflecting upon the enunciation, endeavour to discover in the relations established among the numbers which enter it, the succession of operations to be performed upon the known quantities, in order to deduce from them the values of the unknown.*

Problem First—*Required, a number, of which the half, third, fourth, and* $\frac{2}{7}$*ths, added together, form the number* 575?

We commence by remarking that, to take the $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{2}{7}$, of any number, and add them together, is the same thing as multiplying this number by the sum of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{2}{7}$, or by $\frac{115}{84}$. Now, since the product of the number sought by $\frac{115}{84}$ must be equal to 575, it results from the definition of division, that this required number is equal to the quotient of 575 divided by $\frac{115}{84}$; and consequently equal to $575 \times \frac{84}{115}$.

Performing the operations indicated, we find 420 for the number.

Verification, 420
The half, = 210
One-third, = 140
One-fourth, = 105
One-seventh, = 60
One-seventh, = 60
Total, 575

Problem Second.—Required, three numbers whose sum is equal to 96, *and such that the second exceeds the first by* 2, *and the third exceeds the sum of the other two by* 4.

It is evident that, if we diminished the second number by 2, it would become equal to the first; and that if we diminished the third by 2 + 4, or by 6 units, it would become equal to double the first; thus, the sum of the three numbers would be, after these two subtractions, four times the first number.

Now, the difference between 96 and 2 + 4 + 2, or 8, is 88; whence, we see, that the first number is

equal to one-fourth of . . . . 88 = 22
Then the second is . . . . . 22 + 2 = 24
And the third . . . . . . 22 × 2 + 6 = 50
Verification, 96

Problem Third.—Three workmen are employed to do a piece of work; the first could do it alone in 12 *days, working* 10 *hours a day; the second in* 15 *days, working* 6 *hours a day; the third in* 9 *days, working* 8 *hours a day. Required,* 1*st. In what number of hours the three men working together can do the work;* 2d. *What part of it each one will do;* 3d. *How much each one ought to get for his labour, the price of the whole work being* $108?

Solution.—We observe that, according to the enunciation, the first workman could do the work in 12 × 10, or 120 hours; then, in 1 hour, he could do $\frac{1}{120}$ of the work. The second could do it in 15 × 6, or 90 hours; thus, in one hour, he could do $\frac{1}{90}$ of it. The third would do it in 9 × 8, or 72 hours; then in one hour he would do $\frac{1}{72}$ of it. These three workmen labouring together would then, in 1 hour, do

$$\frac{1}{120} + \frac{1}{90} + \frac{1}{72} = \frac{12}{360}, \text{ or } \frac{1}{30} \text{ of the work.}$$

Now, if in one hour they do $\frac{1}{30}$ of the work, they would do the whole in 30 hours.

Again, since in one hour the first workman does $\frac{1}{120}$, in 30 hours, he will do $\frac{1}{120} \times 30$, or $\frac{1}{4} = \frac{3}{12}$. In the same manner, the second, in 30 hours, performs $\frac{1}{90} \times 30$, or $\frac{1}{3} = \frac{4}{12}$. Finally, the third does $\frac{1}{72} \times 30$, or $\frac{5}{12}$.

Then, to find the amount to be paid to each man, we must divide $108 into parts proportional to the three fractions, $\frac{3}{12}$, $\frac{4}{12}$, $\frac{5}{12}$, or the three numbers, 3, 4, 5; which gives $27, $36, and $45, for the respective wages of the labourers.

EXERCISES.

1. A vessel has provisions for only 19 days; yet, by calculations, 25 days must elapse before she can reach a port. Required, how much the ordinary rations must be reduced?

2. Twenty workmen, working 15 days, 10 hours a day, excavated a ditch 65 yards long, by 2·30 yards wide, and ·75 of a yard deep. Required, how many days it would take 36 men, working 12 hours a day, to dig a ditch 200 yards long, by 3 yards wide, by 1·25 yards deep; the difficulty of excavating the first earth being to that of the second as 3 to 4.

3. For what period must $3000 be placed out at interest at 6 per cent. per annum, in order to bring $1325·50?

4. What is the rate of discount on a note of $2500, payable in 18 months, for which the sum of $1860·45 was paid in cash?

5. Four partners invested the same sum in an enterprise; the funds of the first were in the business for 8 months; the

second for 7 months; the third for 10 months; and the fourth for 1 year. Divide the profit of $1800 proportionally to the investments augmented by the interest on each, at the rate of 4 per cent. per annum?

6. We wish to divide $60,000 among three persons, so that the second shall have twice as much as the first, less $2500; that the third shall have three times as much as the first, less $5000. What is the share of each person?

7. Two pounds of copper, at 45 cents; 7 pounds of zinc, at 70 cents; 9 pounds of antimony, at 50 cents, are melted together. What is the price of one pound of the alloy?

8. A person was asked how much money he had in his purse. He answered, If you add to the sum which I have, $\frac{1}{3}$, $\frac{2}{7}$, and $\frac{3}{4}$ of that sum, I would then have 175 dollars. What sum of money has he?

EXAMPLES.

For the convenience of teachers, we annex the following examples for practice, as but few are embodied in the work itself. These are chiefly selected from different practical compilations on arithmetic.

Addition.

Add together, 1225, 3473, 7581, 9064, and 6060. *Ans.*
Add together, 3004, 523, 8710, 6345, and 784. *Ans.* 19366.
Add together, 7500, 234, 646, and 19760. *Ans.* 28140.
Add together, 182796, 143274, 32160, 47047. *Ans.* 405277.
Add together, 66947, 46742, and 132684. *Ans.* 246373.

Subtraction.

16844	103034	5987432
9786	69845	278459
7058	33189	
7896600	5403257	5789232
5403257	4250268	410204

Multiplication.

1st. Multiply 328 by 2. *Ans.* 756.
Multiply 745 by 3. *Ans.* 2235.
Multiply 20508 by 5. *Ans.* 102540.
Multiply 3605023 by 6. *Ans.* ———
Multiply 9097030 by 9. *Ans.* ———

2d. Multiply 725 by 300. *Ans.* 217500.
Multiply 35012 by 2000. *Ans.* 70024000.
Multiply 9120400 by 90. *Ans.* 820836000.
Multiply 4890000 by 36000. *Ans.* ————

3d. Multiply 793 by 345. *Ans.* 273585.
Multiply 471493475 by 4395. *Ans.* 2072213822625.
Multiply 89999000 by 97770400. *Ans.* ————————.
Multiply 17204774 by 125. *Ans.* 2150596750.
Multiply 3768 by 4230. *Ans.* ——————.
Multiply 9648 by 5137. *Ans.* 49561776.

Division.

1st. Divide 3788 by 2. *Ans.* 1894.
Divide 4736511 by 9. *Ans.* 526279.
Divide 78920 by 5. *Ans.* 15784.
Divide 364251 by 3. *Ans.* 121417.
Divide 34300 by 7. *Ans.* 4900.
2d. Divide 1203033 by 3679. *Ans.* 327.
Divide 49561766 by 5137 *Ans.* 9648.
Divide 2150596750 by 125. *Ans.* 17204774.
Divide 71900715708 by 57149.
Ans. 1258127. *Rem.* 15785.
Divide 78674 by 200. *Ans.* 393 + 74 *Rem.*
Divide 32500000 by 520. *Ans.* 62500.
Divide 36000000 by 3600. *Ans.* ———.
Divide 27489000 by 350. *Ans.* 7854.

Vulgar Fractions.

Reduction of Vulgar Fractions to a Common Denominator.

Reduce $\frac{3}{4}$ and $\frac{5}{9}$ to a common denominator.
Ans. $\frac{27}{36}$, $\frac{20}{36}$.

Reduce $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$ to a common denominator.
Ans. $\frac{20}{60}$, $\frac{45}{60}$, $\frac{48}{60}$.

Reduce $\frac{2}{10}$, $\frac{3}{5}$, $\frac{4}{7}$, and $\frac{5}{9}$, to a common denominator.
Ans. $\frac{630}{3150}$, $\frac{1890}{3150}$, $\frac{1800}{3150}$, $\frac{1750}{3150}$.

Reduce $\frac{47}{6}$, $\frac{4}{9}$, $\frac{3}{12}$, and $\frac{2}{18}$, to a common denominator.
Ans. ——————.

Finding the Least Common Multiple.

Find the least common multiple of 13, 12, and 4.

Ans. 156.

What is the least common multiple of 11, 17, 19, 21, and 7?

Ans. ———.

Find least common multiple of 6, 9, 4, 14, and 16.

Ans. 1008.

What is the least common multiple of 1, 2, 3, 4, 5, 6, 7, 8, 9?

Ans. 2520.

Reduction of Fractions to the Least Common Denominator.

Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$, to the least common denominator.

Ans. $\frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{12}$, $\frac{10}{12}$.

Reduce $\frac{3}{15}$, $\frac{4}{24}$, and $\frac{8}{9}$, to the least common denominator.

Ans. $\frac{72}{360}$, $\frac{60}{360}$, $\frac{320}{360}$.

Reduce $\frac{1}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{16}$, and $\frac{17}{24}$, to the least common denominator.

Ans. $\frac{16}{48}$, $\frac{36}{48}$, $\frac{40}{48}$, $\frac{42}{48}$, $\frac{33}{48}$, $\frac{34}{48}$.

Reduce $\frac{5}{6}$, $\frac{8}{9}$, $\frac{7}{22}$, $\frac{3}{4}$, and $\frac{5}{12}$, to the least common denominator.

Ans. ——————.

Reduce $\frac{2}{3}$, $\frac{3}{2}$, $\frac{4}{9}$, $\frac{2}{12}$, $\frac{5}{4}$, to the least common denominator.

Ans. ——————.

Greatest Common Divisor.

Find the greatest common divisor of 24 and 36.

Ans. 12.

Find the greatest common divisor of 312 and 504.

Ans. 24.

Find the greatest common divisor of 9024 and 3760.

Ans. 752.

Find the greatest common divisor of 4410 and 5670.

Ans. 630.

Find the greatest common divisor of 3775 and 1000.

Ans. ——.

Find the greatest common divisor of 101 and 859.

Ans. ——.

Reduction of Fractions to their Simplest Terms.

Reduce $\frac{512}{4096}$ to its simplest terms.	*Ans.* ——.
Reduce $\frac{160}{168}$ to its simplest terms.	*Ans.* $\frac{20}{21}$.
Reduce $\frac{2640}{2880}$ to its simplest terms.	*Ans.* $\frac{11}{12}$.
Reduce $\frac{549}{7143}$ to its lowest terms.	*Ans.* $\frac{183}{2381}$.
Reduce $\frac{63}{81}$ to its lowest terms.	*Ans.* ——.
Reduce $\frac{410}{510}$ to its lowest terms.	*Ans.* ——.

Addition of Fractions.

Add $\frac{3}{7}$, $\frac{4}{2}$, and $\frac{3}{5}$ together.	*Ans.* $\frac{106}{35}$.
Add $\frac{4}{5}$, $\frac{6}{5}$, and $\frac{33}{5}$ together.	*Ans.* ——.
Add $\frac{4}{16}$, $\frac{12}{7}$, 1, and $\frac{16}{9}$ together.	*Ans.* ——.
Add $\frac{6}{12}$, $\frac{3}{5}$, $\frac{4}{8}$, and $\frac{6}{30}$ together.	*Ans.* ——.
Add $\frac{1}{16}$, $\frac{3}{7}$, $\frac{2}{8}$, and $\frac{4}{9}$ together.	*Ans.* $\frac{1195}{1008}$.

Conversion of Fractions into Mixed or Entire Numbers, and vice versâ.

Reduce $\frac{6}{3}$ to its equivalent whole number.	*Ans.*
Reduce $\frac{7}{2}$ to a mixed number.	*Ans.* $3\frac{1}{2}$.
Reduce $\frac{15}{4}$ to a mixed number.	*Ans.* $3\frac{3}{4}$.
Get out the entire part of $\frac{482}{20}$.	*Ans.* $24\frac{2}{20}$.
Get out the entire number in $\frac{97}{8}$.	*Ans.* $12\frac{1}{8}$.
Find the entire part in $\frac{4790}{25}$.	*Ans.* ——.
Reduce $\frac{1512}{108}$ to a whole or mixed number.	*Ans.* ——.
Bring $144\frac{5}{9}$ to a fractional form.	*Ans.*
Bring $47\frac{5}{6}$ to a fractional form.	*Ans.*
Reduce $31\frac{7}{10}$ to a fractional form.	*Ans.* $\frac{317}{10}$.
Add $\frac{75}{8}$, $\frac{4}{9}$, $3\frac{2}{3}$, $4\frac{5}{6}$ together.	*Ans.*
Add $6\frac{5}{12}$, $2\frac{1}{3}$, $4\frac{3}{4}$ together.	*Ans.*

Subtraction of Fractions.

Subtract $\frac{1}{6} + \frac{1}{4}$ from $\frac{1}{2} + \frac{1}{3}$.	*Ans.*
From $\frac{4}{6}$ take $\frac{2}{6}$.	*Ans.*
From $5\frac{3}{8}$ take $4\frac{1}{3} + \frac{3}{4}$.	*Ans.*
From $\frac{314}{549}$ take $\frac{1}{25}$.	*Ans.*

Multiplication of Fractions.

1st. Multiply $\frac{1}{7}$ by 8.
Multiply $\frac{2}{3}$ by 5.
Multiply $4\frac{1}{2}$ by 3.
Multiply $\frac{3}{8}$ by 24.

2d. Multiply 7 by $\frac{1}{6}$.
Multiply 22 by $\frac{1}{3}$.
Multiply 15 by $\frac{4}{3}$.

3d. Multiply $\frac{5}{6}$ by $\frac{2}{3}$ by $\frac{5}{7}$.
Multiply $3\frac{2}{7}$ by $4\frac{14}{33}$.
Required the product of 5, $\frac{2}{3}$, $\frac{2}{7}$, $\frac{3}{5}$, and $\frac{4}{5}$.
Required the product of $4\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{7}$, and $18\frac{4}{5}$.

Division of Fractions.

1st. Divide $\frac{18}{37}$ by 9.
Divide $\frac{24}{27}$ by 7.
Divide $\frac{61}{21}$ by 37.

2d. Divide 10 by $\frac{2}{7}$.
Divide 7 by $\frac{32}{12}$.
Divide 28 by $\frac{14}{15}$.
Divide 16 by $\frac{8}{22}$.

3d. Divide $\frac{1}{8}$ by $\frac{1}{7}$.
Divide $4\frac{1}{2}$ by $2\frac{3}{4}$.
Divide $\frac{64}{111}$ by $\frac{23}{1213}$.
Divide $371\frac{1}{2}$ by $\frac{1}{414}$.

Fractions with Fractional Terms, or Complex Fractions.

Reduce $\dfrac{4\frac{1}{2}+\frac{2}{3}}{3\frac{2}{7}+\frac{1}{2}}$ to a simple fraction.

Add $\dfrac{4\frac{7}{8}-\frac{1}{2}}{2\frac{3}{4}+\frac{1}{3}}$ to $\dfrac{5\frac{1}{2}}{6\frac{3}{4}}$.

From $\dfrac{4\frac{1}{3}+\frac{2}{7}}{25+5\frac{1}{2}}$ take $\dfrac{\frac{4}{7}}{\frac{3}{8}}$.

Multiply $\dfrac{27\frac{3}{4}}{\frac{2}{3}}$ by $\dfrac{\frac{4}{5}}{2\frac{1}{2}}$.

Divide $\dfrac{8\frac{1}{2}+\frac{1}{4}}{2\frac{3}{4}-\frac{1}{3}}$ by $4\frac{3}{4}$.

Fractions of Fractions; or, Compound Fractions.

Reduce $\frac{1}{2}$ of $\frac{5}{3}$ of $\frac{4}{7}$ to a single fraction.
Reduce $\frac{1}{2}$ of $\frac{5}{6}$ of $\frac{1}{9}$, or $\frac{2}{3}$ of 20 to a single fraction.
Reduce $\frac{1}{4}$ of $\frac{1}{12}$ of $\frac{1}{20}$ to a single fraction.
Add $\frac{1}{3}$ of $\frac{20}{7}$ to $\frac{2}{5}$ of $\frac{5}{6}$.
Multiply $\frac{1}{2}$ of $\frac{7}{8}$ by $\frac{27}{29}$ of 2.

Approximate Valuation of Fractions by means of Fractions having Smaller Terms.

Value $\frac{593}{619}$ in twelfths.
Find the approximate value ot $\frac{269}{331}$ in ninths.
Find the approximate value of $\frac{521}{683}$ in eighths.
Find the approximate value of $\frac{643}{854}$ in tenths, in hundredths, in thousands.

Denominate Numbers.

Reduction of Compound Numbers.

1st. Reduce 59 ℔. 13 *dwt.* 5 *gr.*, to grains.
Reduce £121 0*s.* 9½*d.*, to half pence.
Reduce 365 days, 5*h.*, 48′, 48″, to seconds.
Reduce 5 miles, 3 furlongs, 1 pole, and 2 yards, to yards.
Reduce 375 *cwt.* 292 *lb.* 5 *oz.*, to ounces.
Reduce 77 *A.* 1 *R.* 14 *P.*, to perches.

2d. Find the compound number of pounds, shillings, and pence, in 5900 pence.

Find the number of tons, cwts., &c., in 36325 *lbs.*

Convert 123200 yards into a compound number, whose principal unit is miles.

How many pounds, ounces, &c., in 704121 grains?

In 5927050 minutes, how many weeks, days, &c.?

3d. Reduce 25 days, 3 hours, 5 minutes, to the fraction of a year.

Reduce 3 furlongs, 2 poles, 3 yards, and 2 feet, to the fraction of a mile.

Reduce 10 *lbs.*, 12 *oz.*, to the fraction of a *cwt.*

Reduce $\frac{1}{2}$ a penny to the fraction of a pound.

Reduce $4\frac{1}{2}$ grains to the fraction of a pound Troy.

4th. Convert $\frac{2}{3}$ of a pound into shillings and pence.

Convert $\frac{7}{26}$ of a year into days, hours, &c.

Convert $\frac{3}{8}$ of a mile into a compound number of its lower denominations.

Convert $\frac{4}{5}$ of a pound Troy into *oz.*, *dwts.*, *grs.*

Convert $\frac{5}{16}$ of a *cwt.* into its equivalent compound number.

Addition of Compound Numbers.

Form the following sums:

(1)			(2)			(3)		
£.	*s.*	*d.*	*yds.*	*ft.*	*inches.*	℔.	*oz.*	*drs.*
149	14	$7\frac{1}{4}$	5	2	$6\frac{1}{2}$	17	15	15
37	11	$9\frac{3}{4}$	4	1	$2\frac{3}{4}$	27	14	11
69	14	7	31	0	$10\frac{3}{4}$	16	13	9
64	13	10	6	1	11	70	0	0
47	14	$10\frac{1}{4}$	51	2	5	5	6	$7\frac{1}{2}$

Add $\frac{1}{2}$ £ and $\frac{2}{3}$ of a shilling together.

Add $\frac{2}{3}$ *cwt.* and $\frac{1}{8}$ of an ounce together.

Add $\frac{1}{2}$ mile, $\frac{2}{3}$ of a furlong, $\frac{3}{4}$ of a yard together.

Subtraction of Compound Numbers.

	(1)			(2)			(3)	
£.	*s.*	*d.*	℔.	*oz.*	*dr.*	*fur.*	*rod.*	*yd.*
5	3	10	125	0	10	13	34	3¾
2	10	11	27	1	15	12	39	5½

From 2 £ take 10 pence.

From ½ yard take ¼ inch.

From ⅚ of a *lb.* take 2½ grains.

Multiplication of Compound Numbers.

Multiply £1 11*s.* 6*d.* by 5.

Multiply £1 17*s.* 6*d.* by 63.

What is the cost of 9 *cwt.* 5 *lbs.* of sugar, at £1 11*s.* 5*d.* per *cwt.?*

What is the cost of 7 *yds.* 2 *ft.* 3 *in.* of cloth, at the price of £3 6*s.* 4*d.* per yard?

Multiply 5 feet 6 inches by 10 feet 10 inches.

Multiply 7 yds. 2 feet 3 inches by 11 feet 10 inches.

What is the cost of ¾ yard of cloth, at ¾ £ per yard?

What is the cost of 2 *lbs.* ½ *oz.* of a commodity which costs 2*s.* ½*d.* per *lb.?*

Division of Compound Numbers.

Divide £69 11*s.* 9*d.* by 9.

Divide £28 2*s.* 1½*d.* by 6.

Divide 375 miles, 2 fur., 7 poles, 2 yds., 1 foot, 2 in., by 39.

If 9½ yards of cloth cost £4 3*s.* 7½*d.*, what is the price per yard?

A man's income is £140 a year; what is it per diem?

If 2½ yards of cloth cost 10½*s.*, what will ¾ of a yard cost?

If 66 *lbs.* of sugar cost £4 2*s.* 4*d.*, what is the price per *lb.?*

Divide 126 square feet by 2 feet 10 inches.

Decimal Fractions.

Numeration of Decimals.

Write in figures the following numbers:—

Eight, and two thousand seven hundred and seven ten millionths.

Forty-five, and seventeen hundred thousandths.

One, and one hundred and one billionths.

Twenty-four, and forty-five ten thousandths.

Five thousand six hundred and eighty-two, and two ten millionths.

Five thousand and one ten millionths.

Five hundred and one ten thousandths.

Addition of Decimals.

Add 0·1257, 257·00101, 3256·05, 22·056, 3·25, 2·207, and 0·002256 together.

Add 0·0009, 1·0436, 3, 0·02, and ·028 together.

Add 3·0739, 5867, 0·00000201, 25·06, 0·6, 0·21, 1·75, and ·003 together.

Add 28·29, 2·829, 0·2829, 311212105·6, 3112, ·121056, 4·0003, and ·01 together.

Subtraction of Decimals.

From 27·06, subtract 2·05078.

From 36·055, take 0·072530.

From 9, take ·9, ·09, ·009, and ·0009, in succession.

From 10·00001, take 0·11111112.

From 27·854, take 25·9999.

Multiplication of Decimals.

Multiply ·573005 by ·000754.

Multiply 2·01013 by 24.

Multiply ·356 by 12000.

Multiply ·55 by ·55.

Multiply 3·00001 by ·00002.

What is the price of 52·756 yds. of cloth, at 10·06 shillings per yard?

What is the compound number of *lbs. oz.*, &c., in ·625 of a cwt.?

How many shillings and pence in ·3333 of a £?

Convert ·076 of a mile into yards, &c.

Convert ·04678 of a pound avoirdupois into *oz.* and *dr.*

Division of Decimals.

Divide 11·8652 by 2·303.

Divide 34·77421 by 1·03.

Divide ·0100001 by ·01.

Divide 22·0784 by ·002.

Divide 475·28677 by ·4, by ·04, by ·004, by ·0004, by ·00004.

Divide 1572·36620 by 980.

Write the respective quotients of 28·79 by 10, 100, 1000, 10000, 100000, ·1, ·01, ·001, ·0001, ·00001.

Divide ·1 by ·0001.

Divide 9 by ·9, by ·09, by ·009, by ·005, by ·00012.

Convert ·122 of a shilling into the decimal of a £.

Convert 0·98 of a *lb.* avoirdupois into the decimal of a *cwt.*

Divide 94·0369 by 81·022.

Conversion of Vulgar Fractions into Decimals, and some Miscellaneous Examples.

Reduce $\frac{6}{7}$ to decimals.

Reduce $\frac{17}{19}$ to decimals.

Reduce $\frac{10}{12}$ to decimals.

Reduce $\frac{36}{1280}$ to decimals.

Divide 10 by 563 into five decimal places.

Convert the decimals 0·75, 0·25, 0·5, and 0·225 to their simplest form in vulgar fractions.

Convert 10*s.* 6*d.* to the decimal of a £.

Convert $9\frac{1}{2}$ months into the decimal of a year.

Convert 17 *hrs.* 10 *min.* 25 *sec.* to the decimal of a day.

What is the compound number in £5·75?

Decimal Denominate Numbers.

What is the price of 82·125 metres of cloth, at $6·76 per metre.

Multiply 89·767 metres by 2·25 metres.

Divide 66·787 square metres by 10·375 metres.

Convert 53·84 metres into feet.

Convert 520·687 grammes into *lbs.* avoirdupois.

Convert 15 feet 6 inches into metres.

Convert 25 *lbs.* 6 *oz.* avoirdupois into grammes.

Convert 25° 36′ 56″ of the sexagesimal division of the circle into its equivalent in the centesimal.

Convert 209° Fahrenheit's thermometer into its equivalent on the Centigrade.

Different Systems of Numeration.

Convert 325 and 422 of the decimal system into their equivalent numbers in the nonary system, and multiply them together in that system.

Convert 101, 233, 22101, of the quaternary system, into their equivalent numbers in the system whose base is six.

Multiply 3023 by 4012 in the quinary system, and convert the result into its equivalent in the decimal system.

Add 10011, 1001110, 101101, 101111 of the binary system, and convert the result into its equivalent in the decimal system.

All the Divisors of a Number.

Find all the divisors of 2820. Number of divisors, 24.
Find all the divisors of 38088. Number of divisors, 36.
Find the divisors of 1764.
Its factors are $2^2 \times 3^2 \times 7^2$. No. of divisors, 27.
Find the prime divisors of 1665. *Ans.* 3^2, 5, 37.
Find the prime divisors of 56700. *Ans.* $2^2 \times 3^4 \times 5^2 \times 7$.
Find the prime divisors of 122108 *Ans.* $2^2 \times 7^3 \times 89$.
Find the prime divisors of 3329. *Ans.*

Greatest Common Divisor by Prime Factors.

Find the greatest common divisor of 12321 and 54345 by prime factors.

Find the G. C. D. of 3775 and 1000.

Find the G. C. D. of 24720 and 4155.

Greatest Common Divisor of Several Numbers.

Find the G. C. D. of 1260, 1512, 2016, and 7350, by the method of prime factors.

Find the G. C. D. of 492, 744, and 1044.

Find the G. C. D. of 216, 408, and 740.

Periodical or Repeating Decimals.

Convert the vulgar fraction $\frac{5}{7}$ into a periodical decimal.

Convert $\frac{19}{37}$ into a periodical decimal.

Convert $\frac{428}{315}$ into a periodical decimal.

Convert $\frac{19}{650}$ into its periodical decimal.

Find the *generatrix* or vulgar fraction corresponding to the decimal 0·99999 . . . . *Ans.* 1.

Find the generatrix of the repeating decimal, 0·012345679012345679 . . . . *Ans.* $\frac{1}{81}$.

Find the generatrix of 0·98765432098765432O . . . . *Ans.* $\frac{80}{81}$.

Find the vulgar fraction corresponding to the decimal, 8·927783783 . . . . *Ans.* $\frac{41291}{4625}$.

Find the vulgar fraction corresponding to the repeating decimal, 0·36538461538461. *Ans.* $\frac{19}{52}$

Rule of Three.

If $\frac{3}{8}$ of a yard of cloth costs 10*s.* 6*d.*, how many yards can be bought for £13 15*s.* 6*d.*?

If 100 workmen can finish a piece of work in 22 days, how many will it require to finish the same work in 4 days?

If 10 *cwt.* can be carried 54 miles for 27 shillings, how many pounds can be carried 20 miles for the same money?

If 15 yards of stuff, $\frac{3}{4}$ yard wide, cost 27*s*. 6*d*., what will 40 yards of the same stuff cost, one yard wide?

If the keeping a horse costs $87\frac{1}{2}$ cents a day, what will it cost to keep 11 horses for one year?

A man breaks, owing $14,000·57, his property amounting to $7840·26. How much will his creditors receive in the dollar?

Compound Proportion—Reduction to Unity.

If a man travels 150 miles in 4 days, travelling 12 hours a day, in how many days, travelling 11 hours a day, can he travel 375 miles?

If 150 bushels of corn feed 18 horses 75 days, how many days will 87 bushels feed 11 horses?

If 250 men, in 4 days, working 10 hours a day, dig a trench 275 yards long, 3 yards wide, and 2 yards deep, in how many days, working 9 hours a day, will 25 men dig a trench 430 yards long, 4 wide, and 3 deep?

If a regiment of soldiers, consisting of 970 men, consume 350 bushels of wheat in 4 months, how many soldiers will consume 1500 bushels in 3 months, at the same rate?

If the transportation of 15 cwt., 2 quarters, 72 miles, cost $5·64, what will the transportation of 5 cwt., 3 qrs., 112 miles, cost?

Rule of Simple Interest.

What is the *interest* on $8079·74, for 5 years, at 6 per cent?

What is the interest on $3750, at $4\frac{1}{2}$ per cent. for $5\frac{1}{2}$ years?

What is the interest on $3375, for 5 months, at 6 per cent. per annum?

What is the interest on $4500, at 5 per cent. per annum, for 280 days?

What is the interest on $3195·54, for 7 years, 6 months, and 22 days, at 6 per cent. per annum?

What is the interest on £1047 3*s*., for $3\frac{1}{2}$ years, at 5 per cent.?

What is the interest on \$5556·25, for 2 years, 7 months, 21 days, at $4\frac{1}{2}$ per cent. per annum?

In what time will \$500 bear an interest of \$500, at 6 per cent. per annum?

What must be the rate of interest in order that a sum put out at interest must double itself in $16\frac{2}{3}$ years.

What sum put out at interest, at the rate of 6 per cent. per annum, will produce \$575?

Percentage.

(We add a few questions, the solution of which is a simple application of the rule of proportion.)

A man invested \$12,000, and lost 64 *per cent.* of it. How much had he left?

Two men had each \$500. One spends $12\frac{1}{2}$ per cent. of his money; the other 15 per cent. of his. How many more dollars did the last spend than the first?

A merchant laid out \$250 as follows: — He pays 25 per cent. of his money for clothes; 30 per cent. of what is left for sugar; 12 per cent. of what is then left for calicoes. How much had he remaining?

A man has \$750, and spends \$85. What per cent. of his money has he expended?

Out of a cask of 500 gallons, 60 gallons are drawn. What per cent. is this?

If I pay \$756·75 for 5 hogsheads of tobacco, and sell them for \$965·25, what per cent. do I gain on the purchase-money?

Rule of Discount.

(Examples to be worked either by the usual rule, or by the accurate rule of Art. (181).

A. has a note against B. for \$5746, payable in 4 months. He gets it discounted at 7 per cent. per annum. How much does he receive?

A planter sold produce to the amount of $12,574, payable in 6 months. He gets his note discounted at 6 per cent. per annum. How much does he receive?

For what amount must a note be drawn, payable in 1 year, 3 months, and 5 days, so that, when discounted, its present value at 7 per cent. per annum shall be $507·27?

What is the present value of $8250, payable as follows:— One-half in 4 months; one-third in 6 months; the rest in 9 months; the rate of discount being 6 per cent. per annum?

What is the present value of $5000, payable in 9 months, the rate of discount being 4½ per cent. per annum?

I bought goods for $7500 in cash, and sold them for $9000, payable by a note in 6 months. What will be my gain, if I discount the note at 6 per cent. per annum?

Rule of Fellowship.

A. and B. have gained by trading, $230. A. put into stock $300; B. $500. What is each one's share of the profit?

A. and B. have a joint stock of $4200, of which A. owns $3600, and B. $600; they gain in a year $2000. What is each one's share of the profits?

Three merchants, A. B. and C., freight a ship with 4340 tons of coal. A. puts in 1350 tons; B. 875 tons; and C. the rest. In a storm, the seamen were obliged to throw 500 tons overboard. How much of the loss must each merchant sustain?

A. put in trade $500 for 4 months, and B. $600 for 5 months; they gained $240. Divide it between them in the compound ratio of the times and capitals.

Four traders form a company. A puts in $400 for 5 months; B. $700 for 8 months; C. $840 for 6 months; D. $1500 for 10 months. They lose $1000. Divide the loss in the composite ratio of the times and sums invested.

A. put $1500 in trade for 15 months with B., who put in $1000 for 18 months. They gain $800. Divide the gain in the ratio of the two sums invested, increased by the interest for the two periods, at 6 per cent. per annum.

Rule of Alligation.

A grocer mixes 80 gallons of whiskey, at $37\frac{1}{2}$ cents, with 10 gallons of water, costing nothing. What is the price of one gallon of the mixture.

A man employed 500 workmen, 160 of whom receive wages at the rate of \$2 a day; 200 at \$1·75; and 140 at \$1·50. What is the average per diem of each labourer?

A mixture being made of 5 *lb.* of tea, at 6*s.* per *lb.*; 19 *lb.* at 10*s.* 6*d.* per pound; and 15 *lb.* at 4*s.* 9*d.* per pound. What is one pound of it worth?

On a certain day the thermometer indicated the following temperatures:—From 6 A. M. to 10 A. M., 65°; from 10 A. M. to 1 P. M., 76°; from 1 P. M. to 4 P. M., 87°; from 4 P. M. to 6 P. M., 70°. What was the mean temperature of the day?

Some General Questions.

Divide \$2000 among A. B. and C., so that B. may have \$100 more than A., and C. \$70 more than B.

Find two numbers such, that if we add 21 to the first, the resulting sum shall be 5 times the second number; and if we add 21 to the second, the resulting sum shall be three times the first number.

Two men are travelling on the same road, in the same direction; the first is 50 miles ahead of the second. The first travels 25 miles a day; the second 35 miles a day. How many days must elapse before the second shall overtake the first?

The hour and the minute hands of a clock are exactly together, and it is between 4 and 5 o'clock. What o'clock is it exactly?

A reservoir of water has two cocks to supply it; by the first alone it may be filled in 40 minutes; by the second, in 50 minutes; and it has a discharging cock by which it may, when full, be emptied in 25 minutes. Now these three cocks being all left open, in what time will the cistern be filled?

A father devised $\frac{1}{5}$ of his estate to one of his sons; $\frac{1}{3}$ of the remainder to another; and the remainder to his wife. The sons' legacies differed by $500. What did the widow receive?

There is an island 73 miles in circumference, and three pedestrians start together, to travel in the same direction around it. A. goes 5 miles a day; B. 8; and C. 10. In what time will they all come together again?

What number is that from which, if you take $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder add $\frac{7}{16}$ of $\frac{1}{20}$, the sum will be 10

NOTES.

Note A

We have spoken in (135) of the numbers in any system whose base is b, which possess analogous properties to 9 and 11 in the decimal system. These numbers are $b-1$ and $b+1$, and the properties which they enjoy are the following:

1st. When the sum of the figures of any number whatever is divisible by $b-1$, this number is itself divisible by $b-1$.

2d. Any number is divisible by $b+1$, when the difference between the sum of the figures in the odd places, counting from the right, and the sum of the figures in the even places is 0, or a multiple of $b+1$.

3d. The remainder of the division of a number by each one of the two numbers, $b-1$ and $b+1$, is obtained in the first case by the aid of the sum of the figures; and, in the second case, by the difference between the two sums of the figures of the odd places and those of the even places.

We have indicated in (135) how these properties may be proved.

Algebra furnishes also another means of demonstrating these properties, founded on the principles,—

1st. That b^m-1 is always divisible by $b-1$.

2d. That b^m-1 is divisible by $b+1$, when m is an even number, and b^m+1 is divisible by $b+1$ when m is odd.

We give here the systems of numeration of the Greeks and Romans, with their notation, the latter being still used to indicate ordinal numbers at the beginning of Chapters, Sections, &c.

In the Roman Notation.

One was written with a single mark	I.
Two .... with two marks	II.
Three .... with three marks	III.
Four	IIII.
Five was written	IIIII.
And so on, to ten.	
Ten was written	X.
Two tens	XX.
Three tens	XXX.

And so on, to ten tens, the intermediate number being written by combining the two sets of characters. This being too cumbrous, instead of writing five marks for the number five, they took the upper half of the ten (X) or (V), to express it. And also the convention was adopted (in addition to the one adopted above, viz., that like characters placed together indicate that the numbers thereby represented are to be added) that a character representing a smaller number, placed before a character representing a larger number, indicated that the first was to be subtracted from the second, and placed after it; or on the right, is to be added to it. Thus, V being five, IV would be four, and VI six, VII seven, IX nine, XI eleven, &c.

Instead of writing ten X's for one hundred, the character [was adopted; the lower half of this, L, represented fifty. Then, instead of writing four X's for forty, it is written XL; that is, fifty less ten; and sixty is written LX; seventy, LXX; eighty, LXXX; but ninety, X[. These characters were afterwards replaced by the letters which they resembled. I was put for 1; V for V; X for X; L for L; C for [; D for the character representing five hundred; and M for one thousand.

The fundamental operations upon numbers can be performed very readily by the Roman notation, though the notation is by no means so simple and convenient as the Arabic.

Example. — Add

$$\begin{array}{r}\text{CCLXVIII.}\\ \text{DCLXXIII.}\\ \text{CXLVII.}\\ \hline \text{MLXXXVIII.}\end{array}$$

It is obviously most convenient to begin on the right. Adding the I's, we find eight of them, or one V to be carried, and III remaining. We set down the III, and add the V to those in given numbers. We thus obtain three V's, or one X and one V remaining, which we set down. We add the X to those in the given numbers, which come after the L's, taking care to subtract from this sum according to the convention of the notation, the X, which, in the third number, comes before L. We thus obtain three X's, which, as they do not make one of the denomination of L, we write down in the result. Adding the L's we obtain one C and one L, which we write down. Summing up the C's, we get five C's or one D, which we carry to the next column; and, by adding the D's, we get M, or ten hundreds. The result is then MLXXXVIII. And so on for the other operations.

The Greeks used their letters in several different ways to denote the different numbers. The most general system of notation was the following: — To express the 9 units, 9 tens and 9 hundreds. They divided the alphabet into three parts; but, as the alphabet contained only 24 letters, three new signs were introduced, ϛ′ for six, ϟ for 90, and ϡ for 900. All the numbers less than 1000 were denoted by these letters and signs, with a small mark a little to the right above them. A similar mark under the letter represented thousands. Placing one letter after another indicates that they are to be added together. Thus, $\alpha'=1$, $\iota'=10$, $\beta'=2$, $\alpha_{,}=1000$, $\iota_{,}=10\cdot000$, $\beta'_{,}=2000$, $\iota'\alpha'=11$, $\iota'\beta'=12$, $\varkappa'=20$, $\varkappa'\alpha'=21$, $\varkappa_{,}=20{,}000$, $\rho'=100$, $\rho_{,}=100{,}000$, &c., &c., &c.

Note B.

Abridged Methods of Multiplication and Division.

Questions often arise which require the multiplication of two numbers containing a considerable number of decimal figures, while we wish only to regard a small number of the decimal figures of the product. It is important, then, to have a method of obtaining the product of any two decimals with the degree of approximation which the enunciation of the question requires, without being obliged to calculate all the partial products which the usual mode of multiplication renders necessary.

This method is the *abridged method*, which we will now explain.

Let it be required, for example, to obtain to within less than *one thousandth* the product of the two numbers

84·0783647 and 72·46538.

We would attain the end proposed, if we could form a number which should contain all the thousandths and units of higher order contained in the total product. This we accomplish in the following manner :

Operation Proposed.	*Verification of the Operation.*
84·078364~~7~~	72465380
8356427	~~7~~46387048
588548548	579723040
16815672	28986152
3363132	1507257
504468	57972
42035	2173
2520	434
672	28
6092·770~~47~~	4
	6092·770~~00~~

We commence by placing the multiplier under the multiplicand in *an inverse order*, and so that the figure 2, in the *simple units* place, shall fall under the figure of *hundred-thousandths;* that is to say, under the figure holding the second place on the right of that whose order indicates the degree of approximation required: then the figure 4 of its *tenths* falls under the figure of *ten-thousandths* of the multiplicand; the figure 6 of its *hundredths* under the figure of *thousandths;* and so on for the rest.

By this arrangement each figure of the multiplier corresponds to such a figure of the multiplicand, that the product of the two gives *hundred-thousandths.* Thus, the figure 7 of the *tens* of the multiplier corresponds to the figure of *millionths* of the multiplicand, and their product gives *hundred-thousandths.* In the same manner, if there were *hundreds* in the multiplier, the figure should be placed under the figure of *ten-millionths* of the multiplicand.

Having arranged the numbers thus, we multiply successively all the figures of the multiplicand, beginning on the right by each one of the figures of the multiplier, not taking any account of the figures of the multiplicand, which are situated on the right of the figure by which we are multiplying; and we place the products (considered as resulting from the multiplication of two entire numbers), one under the other, so that their *simple units* shall fall under each other. We add then all these products, and separate on the right of the sum five decimal figures, and draw a mark across the last two.

The part on the left of these last two figures is the required product.

In order to see the reason of this mode of operating, and to convince ourselves that we obtain thus the desired degree of approximation, it suffices to remark that, at each partial multiplication, we neglect several *hundred-thousandths*, the summation of which gives some *ten-thousandths* of error. But, admitting as an average, that we commit an error of 5 hundred-thousandths at each partial multiplication, we see that it would require 10 partial multiplications, or 10 figures in the multiplier, in order that the error might reach 50 *hundred-thousandths,* or 5 *ten-thousandths;* and 20 figures in the multiplier, in order that the error should be 10 *ten-thousandths,* or *one thousandth.*

Verification of the Operation.

In order to verify the result obtained, it is best to pursue the following method: — We take the multiplier for the multiplicand, and reciprocally, as the table of operation shows; and we average the new multiplier as in the first operation; we then perform the partial multiplications in the same manner, except some modifications, which it is necessary to indicate.

1st. We have placed a 0 on the right of the new multiplicand, in order that the last figure, 8, of the new multiplier, may have its correspondent.

2d. We have drawn a line across the first figure, 7, on the left of this multiplier, as it ought not to give any partial product according to the rule established above.

3d. In each partial multiplication, we take care to add to the product of each figure of the multiplier, by the figure above which corresponds to it immediately, the *figure to be carried,* which is the product of this same figure of the multiplier by the figure which is on its right in the multiplicand.

Thus, in the multiplication by the 4th figure, 7, of the multiplier, we have added to the product of 5 by 7 the 2 units which would have to be carried up from the product of the 3 immediately on the right of the figure 5 by the same figure 7.

4th. Finally, arrived at the figure, 7, of the multiplier, across which we have drawn a line, we have multiplied it mentally by the figure 7, which is on its right in the multiplicand, and we have written the figure, 4, to be *carried,* of this mental product, below the preceding product.

This last modification offers two advantages: the first, that it lessens much the errors committed; and the second, that it enables us to judge whether it is necessary to alter by a unit the figure at which the approximation stops, in order to obtain a more exact result.

In the example just discussed, we have found 47 for the last two figures of the first operation, while in the verification we find 60. All the other figures are the same in both operations.

Then, 6092·771 is the value of the required product to within less than a thousandth.

We give an example a little more complicated. Let the two numbers be

1307·510300896472 and 256·10978641,

of which we wish to obtain the product to within less than 0·00001.

Operation.	*Verification.*
1307·510300896$\cancel{4}\cancel{7}\cancel{2}$	256·1097864100
14687901652	$\cancel{2}\cancel{7}\cancel{4}$6980030157031
2615020601792	2561097864100
653755150445	768329359230
78450618048	17927685048
1307510300	1280548932
117675927	25610978
9152570	768329
1046008	2048
78450	230
5228	15
130	334866·18389$\cancel{1}\cancel{0}$
334866·1838$\cancel{0}\cancel{8}$	

The result required is here 334866·18389, to within less than 0·00001.

Example.—Find to within less than 0·01 the product of the two numbers 89·91666 and 47·19.

Remark.—The method can be applied equally well to the approximate multiplication of two entire numbers.

Example. — Required the product of 470256497 by 2305687, to within less than a *million.*

$$\begin{array}{r} 470256497{\cdot}00 \\ 7865032 \\ \hline 94051299400 \\ 14107694910 \\ 235128245 \\ 28215384 \\ 3762058 \\ 329175 \\ \hline 1084264291\cancel{0}\cancel{2} \end{array} \qquad 1084264292 \textit{ millions.}$$

We will certainly obtain the product to within less than a million, by taking account of the *hundreds of thousands* and the *tens of thousands;* that is, by calculating in the product two figures more than the number required. In order to do this, we arrange the multiplier below the multiplicand in an *inverse order*, and so that the figure 7, of its simple units, shall fall under the figure of *tens of thousands* in the multiplicand; then, the *tens* figure 8, of the multiplier, will fall under the thousands figure of the multiplicand, &c. Nevertheless, as the figures in the *hundreds of thousands* and *millions* places would not have corresponding figures in the multiplicand, we supply their places by two 0's annexed.

We can also employ an abridged method of division, when the dividend and divisor are composed of a great number of figures. But, as this method requires, in order to be thoroughly discussed, developments which could not be given here, we will limit ourselves to giving an idea of the mode of operating. We commence by remarking that the process for finding the quotient of the division of two decimal fractions, with a given approximation, can always be reduced to finding the quotient of the division of two entire numbers to within less than unity.

For, let it be proposed, for example, to find the quotient of the division of 1234·569 by 27·35894 to within less than 0·001. According to the rule for the division of decimals, we must place two zeros on the right of the dividend, which reduces the operation to the division of the two numbers,

123456900 and 2735894.

Then, as we wish to obtain the quotient with three decimal places, we place three new zeros on the right of the dividend, and perform the division, taking care only to separate three figures on the right of the quotient for decimals.

The question is then to find the quotient of 123456900000 by 2735894, and only regarding the entire part of the quotient. We are then led to explain the rule to be followed in the abridged division of two entire numbers.

This rule, principally founded upon the fact that, according to the ordinary method, the determination of each one of the figures of the quotient most commonly depends only on the first two or three figures on the left of the dividend; and the first two or three figures on the left of the divisors can be thus enunciated.

Suppress on the right of the dividend as many figures, less two, as there are in the divisor; divide then the part on the left by the divisor, and if there is no remainder, annex to the quotient as many zeros as you have suppressed figures in the divisor.

But if there is a remainder, divide this remainder by the divisor, with the last figure on the right cut off. Nevertheless, in the multiplication of the new divisor by the figure obtained in the quotient, take care to add the figure to be carried, which the product of the figure cut off from the divisor by this figure of the quotient would give.

Divide then the new remainder by the divisor, with its last two figures cut off, and proceed as before.

Continue these successive divisions, suppressing at each division a new figure on the right of the divisor, and stop the operation when the divisor is reduced to its first two figures on the left.

In order to render this method intelligible, we give both the ordinary and abridged method in the table.

Ordinary Method.

$$
\begin{array}{r|l}
540347056789046 & 2786459 \\ \cline{2-2}
\underline{26170115} & 193918897 \\
\underline{10919846} & \\
\underline{25604697} & \\
\underline{5265668} & \\
\underline{24792099} & \\
\underline{25004270} & \\
\underline{27125984} & \\
\underline{20478536} & \\
973323 &
\end{array}
$$

Abridged Method.

$$
\begin{array}{r|r|l}
5403470567 & 89046 & 278\cancel{6}\cancel{4}\cancel{5}\cancel{9} \\ \cline{3-3}
\underline{26170115} & & 1939 \mid 18897 \\
\underline{10919846} & & \\
\underline{25604697} & & \\
\underline{526566} & & \\
\underline{247920} & & \\
\underline{25004} & & \\
\underline{2713} & & \\
\underline{206} & & \\
12 & &
\end{array}
$$

In the second operation, we separate *five* figures on the right of the dividend, since there are *seven* in the divisor; and we divide the part on the left by the divisor, which gives the first four figures of the quotient, 1939, and for the remainder, 526566.

This done, we draw a line across the last figure, 9, of the divisor, and divide 526566 by 278465; the quotient is 1, by which we multiply the divisor, adding 1 to the units figure of the product for the 9 suppressed in the divisor; we then subtract the product from the remainder, and thus obtain the new remainder, 247920. Cutting off a second figure of the divisor, we divide 247920 by 27864, and subtract from the dividend the product of 27864 by the quotient 8, this product being augmented by the 4 *to be carried*, which the multiplication of 8 by the figure 5, which we have cut off, would give. We proceed in the same manner, until we get for the total quotient the number 193918897, the same result which the ordinary method gives.

Let us now apply the process to two decimal fractions, taking the example proposed above, to find the quotient to within less than 0·001.

Ordinary Method.

$$\begin{array}{r|l}
\underline{123456900000} & \underline{2735894} \\
\underline{14021140} & 45124 \\
\underline{3416700} & \\
\underline{6808060} & \\
\underline{13362720} & \\
2419144 &
\end{array}$$

Abridged Method.

$$\begin{array}{r|l|l}
\underline{1234569} & 00000 & \underline{27\cancel{3}\cancel{5}\cancel{9}\cancel{8}\cancel{4}} \\
\underline{140212} & & 45124 \\
\underline{3418} & & \\
\underline{682} & & \\
\underline{135} & & \\
26 & &
\end{array}$$

Having first reduced the operation to that of entire numbers, we remark, that as the seven figures of the dividend, which remain after the suppression of as many figures, less two, as there are in the divisor, do not contain the divisor, it is necessary in the commencement to cut off the last figure, 4, of the dividend, and divide 1234569 by 273589.

We then continue the operation until we obtain 45124, on the right of which we separate three figures for decimals. This gives 45·124 for the required quotient to within less than 0·001.

THE END.

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